13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

THE WORK OF ART : AN EFFECTIVE TEACHING TOOL

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Very present on the sites of the AMS and the MAA, works of art constitute a very effective pedagogical tool. Some of these works and the demonstration of their utility will be treated in this article.

A FEW PEDAGOGICAL SUGGESTIONS

The diversity of teaching institutions, coupled with their internal diversity, local customs and traditions, as well as the diversity of their publics, lead us to tread cautiously concerning various suggestions and recommendations. This banal reminder of prudence and modesty can permit the reader to approach this text with the distance necessary to appreciate the inevitable insufficiences and the specific characteristics corresponding to his/her centers of interest.

The diffusion of a discipline at any level presents technical as well as affective aspects. There is often a tendency to overlook the latter for the former. The projects which I was able to put into place partially take into account the affective aspects. Before discussing these, it would be wise to call the reader's attention to the importance of the concurrent technical aspects needed to reach our objectives, listed as follows:

- to eliminate the psychological barriers separating certain auditors and publics from knowing about the mathematical world, its history and content.
- to introduce some basic elements to this knowledge.
- to instill in people the desire to further explore such knowledge.The objective is thus to practice a form of attractive initiation to mathematics, while bearing in mind that the content and methods must be adapted to the different publics to be addressed.

Concerning the acceptance or rejection of a discipline, the psychological factors inherent to each individual come into play. They are dependent not only on past and present environmental factors, but also on individual physiological capabilities, such as the ability to memorize or to understand, for example. Unfortunately, I don't know of any detailed studies of these phenomena, with which we might be able to better classify and better adapt the contents of the different exposés.

To resume, we should note the extended range of intellectual possibilities of those whose memory is iconographic, geographic, gestural, or auditory, sometimes linked to spontaneous outbursts of empathy, or on the contrary, withdrawal. We must also note the presence or absence of intellectual exercises favorizing the access to and the development of mental activities, such as reciting, memorization, calculating, and explaining, which help to maintain interest.

The connection between the content of a discipline and an individual's needs determine the acceptance or rejection of the said discipline. When both are in harmony, the individual can become addicted to his subject. She/he can become passionate about the subject at hand without necessarily being fully able to define its properties. Thus, the mental mechanisms, the evolution of the neurological process, may or may not spontaneously coincide with the internal mechanisms of mathematical structures, even though these make up the image of natural processes originating primarily from the physical world and deeply embedded in the constitution of all human beings.

The teacher must therefore endeavor to bring out the basic ties between the physical world and the elements constituting the mathematical universe, particularly via the fundamental phenomenon of stability, the latter being expressed in its repeatability and the universality of forms, images and structures. Addition, for example, ritualizes and symbolizes spontaneous and repeated natural acts, carried out in order to insure its temporal-spatial continuity. After this, it is logical to proceed directly to the idea of an additive structure. It then also becomes necessary to examine the objects in our environment which result from light effects. Practicing geometry is one way to develop a catalogue of the shapes of objects, to examine their properties and to follow their light effects.

We must first search the depths of our brain to show that the practice of mathematics is most natural, so that we can vaunt its merits. It can then become possible to create an emotional experience, coming from the unexpected, surprising our intellect and our senses. Where does the marvelous curiosity come from, which allows us to see that special straight lines issued from the apices of a triangle meet at a common point ? The teacher can awaken a joyous curiosity extending to other subjects, and ending up inexorably upon the stimulating and enriching desire to understand. The explanation can come later.

Rarity, novelty, originality of form and content, the exceptional qualities of its realization, these are the elements by which we qualify a work as beautiful or very beautiful. It strikes the mind, the senses, and marks the memory. For Emile Artin and his friend Henri Cartan, mathematics is an art, albeit an art reserved to a few mathematicians. The specialized technician and the true artist are still worlds apart. Art carries with it prestige, however, and to declare mathematics as an art gives it a certain aura and reputation before which we bow in deference, and which attracts as if it were a brand new bright star in the sky, an immense jewel burning with a thousand fires.

Before developing the formality and rationality gradually acquired through education, human beings have reacted promptly to their environment, whether fixed or moving. Early man approached his novel environment with curiosity and joy, discovering the unexpected, often charged with significance. The new objects and experiences were to be inscribed in his immediate memory. An external motivating explanation would contribute to fixing his attention and longterm memory.

Neither arithmetic, algebraic, analytical, or even category theories can be represented in a concrete manner. This can only be found in the fields of geometry and topology, for these fields begin by defining forms which can in large part be materialized or can become specific objects. The forms inherent to these objects acquire a substantial reality in the usual tri-dimensional space, by plays on light (in the figurative and literal senses) when placed inside well chosen environments which bring out their beauty. Sometimes we can move or change these objects, even touch or handle them. Of course, the longer the visual and physical contact remains, the deeper the features will be inscribed in memory and consciousness.

The teacher should first try to direct the listener's attention to the object's quantitative and qualitative specificities and to their relative disposition, pointing out the types of curves, holes, movement, deformations, or possible methods of construction. In the first phase, the goal is to gain the listener's confidence, both regarding the objects, now somewhat familiar, and regarding the warm and radiant person who is showing her/him previously unknown properties, now become interesting because of their universal character. Formalization, by introducing an adapted symbolism and causal manipulation of these symbols, can come later and even much later, when the first motivating information has matured and become acquired.

The interest in these objects and their explanation will be all the more strong depending on the object's esthetic qualities and the speaker's inspirational qualities. We should then try to choose among the very best artists, who understand how to render visible and accessible the objects emanating from the symbolic world of mathematics.

The teacher-presenter should discreetly point out the beauty of these creations. References to artists of the past supporting her/his exposé could serve as an opportunity to broaden the cultural horizon of her/his public. This has become all the more important, for despite the wealth of possibilities offered by the web, and because of the sheer weight of knowledge necessary to master a discipline, the very notion of a non-superficial general culture is tending to decline among the younger generation. The fine arts, sculpture and music represent our internal and external environment, just as mathematics does. As representational art, they share the same functional property in contributing to our temporal-spatial stability. They also present similar structural and universal characters. In showing the conscious or unconscious pervasion of mathematics in the realization of artistic works since time immemorial, we can point to an at times surprising mathematical presence within the modern works. Of course, this is also a way to introduce the works and the conceptual tools used in their creation, to awaken interest in humanity's past, to introduce knowledge, to open minds and broaden thought, in sum to arouse curiosity.

The interest in history goes much further. To cite certain major actors of the 19th century, such as Goethe or Darwin, "All true understanding is genealogical" (Darwin). I was happy to find this point of view strongly expressed by George Steiner, who wrote some wonderful passages concerning mathematics in his book entitled "Unwritten Books" (2007). I regret that the history of mathematics is almost always and at all levels absent from the mathematics curriculum. Particularly as this would strongly contribute to the interest level of the public, and at the same time give out basic knowledge of some past results, which have now become indispensable for the understanding of the present and future. I would, however, clarify what I mean by the history of a discipline. It is not merely a basic succession of dates and facts. It is also more profoundly about being able to reconstruct their origins, about how these ideas, processes, and inventions were born and took root in their authors' minds. While it seems obvious that detailed reconstructions would be nearly impossible to obtain, attempting to draw them along general lines could be a perilous, but very enriching exercise!

PROJECTS AND REALIZATIONS

The most important project is an architectural project, called the ARPAM project, conceived 25 years ago, supported at its origin by the French Mathematical Society and the French Ministry of Research. It is taken up today by ESMA, the European Society for Mathematics and the Arts www.math-art.eu. It involves implanting on a hilly and wooded land ten small buildings (10 in honor of the Pythagoreans), called follies in architectural terms: both their architecture as their decoration are designed to attract all audiences by their intrinsic beauty. As a part of their originality, they all illustrate mathematical concepts and facts.

The project, is a kind of fragmented museum for mathematics. It has provoked some interest from mathematicians from many countries, in France, Greece and Russia particularly, in Slovenia recently. One may consult the text of the lecture given on it in Athens [7], the Russian article [8], and look at some visualisations [9]. Obviously, it is easier to raise funds to build a stadium than to achieve such a cultural project: a kind of unparalleled artistic and educational tool for popularising mathematics. Establishing a relationship with an enlightened sponsor could facilitate its implementation in any country in the world.

Having the advantage of not requiring substantial funds, exhibitions [10] and conferences [11], [12], [13], mathematical fairy tales [14], fit naturally within the framework of this project. As they rely on artistic material, especially works of art, they are designed to serve diffusion of mathematical knowledge: what they are, what they do, what they contain, what they bring, how they go forward.

They can therefore a priori, according to how they are designed and presented, target all audiences. There are many ways to teach mathematics: they depend largely on objectives that one gives oneself. Two main lines are competing, each with its benefits and disadvantages. The first is "pragmatic", the second is "theoretical". Educational systems embracing pragmatic principles show some flexibility, opposed to systems based on theoretical approaches, which tend to be rather rigid, governed by binding curricula which apply to all educational institutions.

In the first system, one can for instance learn to build polyhedra, to make Euclidean or hyperbolic tilings, without going into the theoretical details of the construction of vector spaces and group theory. One is left with the freedom to emphasize such and such aspect of the theories that comes into play.

In the second system, on the contrary one begins with the theory. Only later can one go on to practical exercises applying the theory. To avoid reducing the time devoted to theory, short but meaningful exercises will be prefered.

I myself am working in the content of the French educational system, which is theoretical and closed. Appealing to works of art for educational purposes within the established curriculum is not provided in the official texts. However progress has been made towards flexibility with the introduction, left to the local educational teams, of new activities - however with a very limited number of hours. In this limited context, I was invited to make introductory lectures accompanied with small exhibitions of works contributing to clarify the content of these presentations, the words, the concepts and the facts stated therein. Listeners of these presentations are then asked to complete a questionnaire to assess the potential impact of these lectures on the idea that the public have of mathematics, on the enriching of their knowledge in the fields of mathematics and art. One would wish to renew these investigations some years later, to estimate the impact these interventions have had on the conceptions, the choices and decisions of listeners.

As part of the Conferences that it organizes, every three years on the average, ESMA yearns to make recent work and reflections by mathematicians and artists in terms of educational use of artworks, whether they be pictorial works (created or not by computers), small sculptures - such as mathematical models-, videos and animations, musical elements. The proceedings of these conference held respectively in 2010 and 2013, have been published by Springer for the first and by Cassini for the second, under the title Mathematics and Modern Art [2], Mathematics and Art III [3]. Both books deserve to be complemented by Mathematics and Art [1], also published by Springer; these are the proceedings of a conference held in 2002 under the auspices of the ARPAM, the father of ESMA.

The next ESMA Conference (September 21-25 2016) http://mathema.si/esma/) will be held in Ljubljana. A lecture will analyze the educational content of two power-points presentations which were partially used during presentations to various groups of children, of ages ranging between 6 and 15. These presentations are respectively entitled "Happy New Year" [11] and « Mathematical Pastry" [12]. The reader who would consider making presentations of this nature and for the same purposes of initiation might have an interest in the description of their contents.

The article attached to that lecture has two parts. The first one provides a list of the most important concepts within natural philosophy and mathematics. They may be presented in the course of the lectures. The concepts of energy, change, and stability stemming from Natural Philosophy, and used by mathematicians, are considered through the Platonic principle that "any object attemps to be in a situation to perpetuate its stability through space and through time».

The prolegomena to mathematical concepts is the notion of representation. The fundamental representation is that given by the shades obtained by projection. They have the advantage of being high quality representations, in that they can maintain the proportions of the objects (as expressed by the Thales theorem under the parallel projection), or more modestly the angular properties (as in the stereographic projection, a conical projection).

As objects of all forms of geometry and topology can be visualized, most mathematical ideas and concepts that appear in the article relate to these theories. The common characteristics of these objects are the presence of singularities, their structure of fiber space, which can be foliated into leaves of varying thickness. It will be noted that the singularities, in many respects highly significant, form rare sets.

One speaks of tiling or tessellation when each thick leave is covered by the same type of patterns, also known as tiles. They may be of equal metric size, or for example, decrease in size uniformly towards the infinitely small, giving rise to what is called a fractal object. These tiles are usually polygons, polytopes, with beautiful symmetries. Note that we can deform a tiling so that the symmetry of the tiles look no longer obvious, they seem hidden.

The concept of symmetry is a kind of corollary of the notion of stability, because it is the balance of internal forces, and thus their symmetrical arrangement, which ensures that stability. The study of the symmetry properties of objects is essential and makes it possible to characterize the intrinsic aspects of their form. The study of how symmetries are organized, structured gave birth to the theory of groups. Each mathematical object is characterized by a plurality of groups of its own. As it is a move that allows to pass from one element of symmetry to another, the symmetry groups are actually groups of motions, the main motions being permutations, translations and rotations. One does not fail to mention the statement of Aristotle-Liouville that any local movement is composed of local translations and local rotations.

These are some essential concepts that are among many others in that article. Ten key features of the presentations are the subject of the second part. I will summarise them briefly.

1 & 2. The first two concern on the one hand the fact that the content of the lectures is in no way subjected to any evaluation of the audience, and on the other hand the atmosphere in which the presentations are given. It aims to be relax, first through exchanges between the audience and the speaker, and second by the topics tackled in a humorous way and filled with novelties for the public.

3. The audiences that these presentations are aimed at, have no idea what mathematics really are. A few simple statements like this:

a somewhat elementary,

but effective way, to describe and represent

essential features of the world around us.

They facilitate the understanding and prediction.

not only implicitly describe the essential part of the activity of the mathematician, emphasizes the importance of the functional characteristics of mathematics from an intellectual point of view and from a social perspective, but also by their apparently banal content are welcome without reluctance or rejection on the part of the audience.

4. In each lecture, one presents, names and promotes new (for the audience) mathematical objects in an active and enjoyable way.

5. Then one directs attention to those elements that are found in any morphology: singularities whose rare but highly significant character one emphasizes and discusses.

A morphology can be fixed or evolving. In other words the best visualization of the latter is achieved by the realization of an animation.

The visual presence of the movement then allows you to view different types of moves, to follow the movements of local shapes along light rays onto the screens on which are drawn the shadows of the projected objects, to see some data characterising the objects which are retained by the projections and allow basically to recognize these objects. It gives also the opportunity to state the Aristotle - Liouville theorem with its disarming simplicity

6. Few profs are given in the context of the presentations for two reasons : firstly a proof takes some time, we could use it more wisely ; on the other hand it is often seen as boring and as such runs counter-productive. But Mathematics are inconceivable without proofs; therefore, we recall during our presentation their function, their scope, their significance related to universal statements. One or two proofs are introduced at this elementary level.

The way we present of these proofs tries to escape from the formal point of view, rather trying to reconstruct and follow the thought processes that could germinate in the minds of those who first have established them.

7. The content of the presentations provides intelligibility of synthetic elements that do not relate only to mathematics but also concern the way of understanding the world, and therefore have an impact on the vision that listeners have of the world and on how they will fit a place in society.

8. Naturally, these statements share common themes, such as the tilings, a special case of thickened foliations. The examination of their constituents, polygons in the case of dimension 2, allows the presentation of the general concept of symmetry, linked to the internal balance and therefore the apparent stability of objects. The study of symmetry groups is beyond the scope of these presentations, though the notion of group is mentioned in connection with the description of elementary movements and vectors.

In general, oral presentations enrich the content that can be read on power-point presentations.

9. Animations ([11], p.23, 37, 54, 56) create soothing moments of rupture in the rhythm of the presentation, but also help to focus attention. They have the interest to synergize several areas, to show almost simultaneously properties or facts apparently related to different specialities. Compared to situations where one fosses on a single property, enrichment can be provided by going back to the animation, therefore showing all its aspects and analyzing the relations that they maintain between them.

10. The last point common to these presentations is of course the presence of bright images, creating a kind of magical spectacle that can leave, with the set of new concepts, indelible marks in memory: yet another educational advantage.

CONCLUSION

A word to conclude on the role of the images in pedagogy.

The power of the image goes far back in the history of mankind, when it shared an animistic worldview. One finds traces of this conception in India for example, where the cow remains a sacred animal, in Tibet through local forms of the Buddhist religion. In ancien animist times, representations of objects were endowed with the same power as the objects. In some cases as in Babylon, numbers were attached with properties of objects and were quite significant. Becoming endowed with power, the image acquires a status symbol, a symbol that sometimes ends up being itself the object of worship.

In other respects images, representations in general, have a rich semantic content complementing the one associated with oral speech, which is often poor, yet to point out some specific aspects of the represented object. The synthesis of these two kinds of semantic occurs in our brains, in the association areas. The presence and operation of these mechanisms make explicit the important role of images in the phenomena of understanding. They make the image into a crucial educational tool.

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