OLD AND NEW MATHEMATICAL MODELS: SAVING THE HERITAGE OF THE INSTITUTE HENRI POINCARÉ

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September 28, 2011

Abstract: We shall give a flavour of the remarkable collection of mathematical models of the Institute Henri Poincaré in Paris by describing some selected pieces.

1. Introduction

During the inaugural conference of the very new European society dedicated to the interaction between mathematics and arts (ESMA), which was held at Institute Henri Poincaré (IHP) in Paris in July 2010, I was given the opportunity to tell about the state of the substantial collection of mathematical objects, which are displayed in the IHP library or are stored in the underground reserve, because of poor condition or which are in process of being cataloged.

This collection has over the century, been subject to various events, including lack of interest, left on shelves or some being moved to Palais de la Découverte (and incidentally photographed by Man Ray in the thirties). However, over the past decade, there has been a renewed interest thanks to those people, who have been encouraged by a more favourable climate after the wave of formalism that had swept through all components of scientific activity, especially in France. After this neglect, we are seeing some beautiful objects, that, abused by some damage of the years, one thinks these are revealing knowledge about the prehistory of geometry.

Although these models have permeated through geometry lessons at all times, students have not always been able to recognize them from their formal description nor from rough sketches on the blackboard.

Thanks to technical progress of computer science, virtual images, in other words, potentially concrete objects, have been able to be portrayed in a realistic way giving rise to a regained attention to two and three dimensional pictures, and, as a result, to a need to have the objects at hand. For the understanding of a mathematical object is greatly facilitated by a back and forth between the necessary abstraction effort and the handling of the physical object, the transition through the computer screen being a go-between no more and no less essential than a stair.

However, there is a problem. The models in Palais de la Découverte and the IHP cannot be handled by visitors at the risk of damage, so they are either locked in cupboards inside the junk room or treated as works of art and placed in showcases. However, unlike works of art, they are intended to be reproduced for their main function is didactic. The main reason for these objects is to give the thought of ordering or making copies, for either personal or student use.

The famous Martin Schilling collection has been sold to many universities at the beginning of the 20th century, and now one can see almost the entire collection at IHP. They have come from the geometry chair of the former Sorbonne and Gaston Darboux used it in his lectures. Most of the wooden models illustrated the descriptive geometry exercises by Joseph Caron at École Normale Supérieure (see for instance [2]). Some models were given in the late forties to IHP by El-Milick, a mathematics professor, who was also a gifted amateur sculptor.

However, this could be said, all this is nothing but past feeling, past mathematics, old-fashioned outdated things. Nevertheless, if the language and the methods evolve, checking a statement on an explicit example is still full of interest, and further, what could be more delightful as to deal with a touchable object able to elicit the geometric sense. In addition, and this is the main point, a collection should not be static but enriched by new models suggested by working mathematicians. Proposing two wire models (a Boy surface of degree six and a Morin surface of degree eight (fig.1)) presently shown in the IHP library, I wished to lead by example.

The virtual issue remains. Today, the computer is present everywhere, from conception to final realization. So, why not stop at the virtual object, since it can be thoroughly explored on a monitor, thanks to more and more efficient 3D software? Wildlife documentaries are no substitute for a zoo.

FIGURE 1. Two wire models realized by the author

It is possible to explore wildlife superficially on the screen, but animals live, become ill, procreate and the zoologist is more concerned with the real animal than its digital glint.

In spite of what a shallow view might let think, a mathematical object, pure abstraction, often stems from real intuition and, as a feedback, can lead to a physical realization that, in addition, especially when it is the unique solution of a natural problem, enjoys plastic qualities that artist souls, like surrealists in the thirties, can take over. One can not refrain from combining visual and tactile pleasure, without depriving oneself of the feeling of substance and materials. Virtual imaging is excellent but it is not all. It is sometimes difficult to develop a correct mental image without having the object to hand.

The full IHP collection consists of about five hundred models that one can roughly classify according to the material and the richness of the class: glass, plastic material, cardboard, wire, wood, sewing thread on a rigid structure, plaster. The Conservatoire National des Arts et Métiers has created its own Museum, why the IHP wouldn't valorize its patrimony? It is now in process. Here, below, there is an example in each class as well as an example of what present technical devices enable one to realize. Pictures of figures 2,3,5,6,7,8,9,10 and 11 are the work of Sabine Starita.

2. The Klein bottle

The famous Klein's surface (fig.2) doesn't exists in our space, that is, in mathematical words, cannot be embedded in the space $\mathbb{R}^3$.

FIGURE 2. Klein bottle

Therefore, we have to do with an immersed image, i.e. with possible self-intersections. As a result, there are several possible models, the actual value is two if we confine our attention to immersed surfaces, in other words, to immersions up to source and target diffeomorphism and a regular homotopy of the target. The one we are speaking about, in glass, justifies well its name of Klein bottle, although according to some historians, its origin would rest on a confusion between the german words "Flasche" and "Fläche".

3. The one-sided cyclid

This model (fig.3) has been realized in July 1947 by Maurice El-Milick and subsequently offered to Paul Belgodère in order to enrich the collection placed in his charge as the librarian of the IHP.

El-Milick produced a number of geometric models, some of them becoming visible on the picture of figure 4. We see noticeably his one-sided cyclid as well as a crosscap in plaster which now belongs to the IHP's collection.

Incidentally, El-Milick one-sided cyclid is not a cyclid, i.e. a surface of degree four doubly passing through the umbilical, but a surface of degree six the shape of which reminding that of the ring Dupin cyclid (fig.5).

FIGURE 3. The one-sided cyclid

It doesn't seem that El-Milick identified his one-sided cyclid as a stable image of the Klein bottle, that is the image of the Klein bottle by a mapping which is not an immersion. Indeed the only singularities are two pinch points, also called Withney umbrellas, connected by a self-intersection line segment. Up to ambient isotopy, the mapping remains unchanged by a small perturbation.

4. THE POINSOT GREAT DODECAHEDRON

Here, we are speaking of a cardboard model coated by a polish (fig.6). It represents the great dodecahedron discovered by Louis Poinsot in 1809.

The edges of the polyhedron are in black. Red lines figure the self-intersection lines of the faces. It has twelve faces, thirty edges and twelve vertices, so that its Euler-Poincaré characteristic is equal to -6, and consequently, since it is orientable, it represents a cell decomposition of the closed orientable surface of genus 4.

FIGURE 4. Maurice El-Milick

5. Smooth cubic with seven real lines

On this wire model of surface realized by Joseph Caron on June 10, 1912, we only see a few noticeable curves traced on it (fig.7).

The surface, the equation of which being

$$Z(X^2 + Y^2 + Z^2) + 2(X^2 - Y^2) - 16Z = 0,$$

is a smooth cubic, that is without any singular point. In 1858 Ludwig Schläfli got the idea to classify such surfaces according to their number of real lines: provided that the cubic surface is not ruled, then it falls into only four different types in reference to the value $3, 7, 15$ or 27 of that number. In the present case, the surface contains seven real lines, six of which being visible, and the seventh being cast aside at infinity. In addition, the surface

FIGURE 5. Ring Dupin cyclid

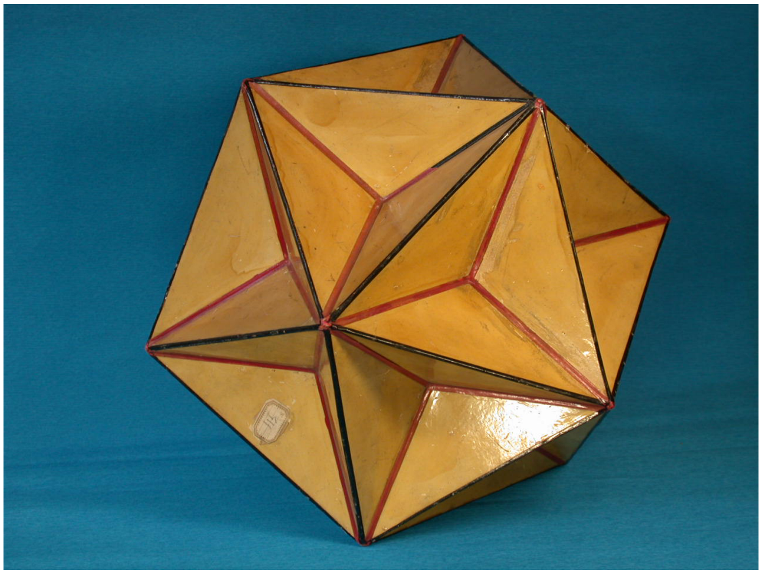

FIGURE 6. The great dodecahedron

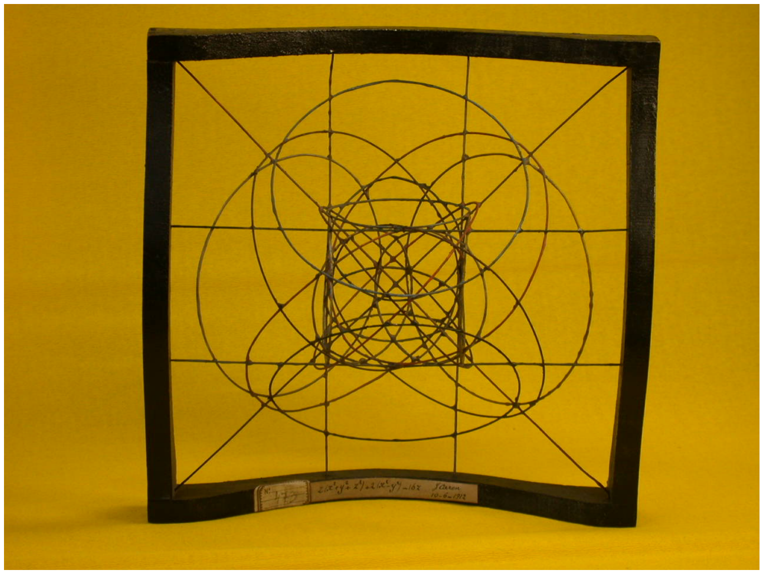

FIGURE 7. Smooth cubic with seven real lines

is generated by ellipses, some of them are circles and are shown on the model. The choice of the curves materialized by the wire preserves the invariance under the group of space isometries leaving the surface globally unchanged, which is isomorphic to the 8-order dihedral group and generated by the two following linear transformations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Although it has a very similar shape (actually it is isotopic through an ambiant isotopy of $\mathbb{R}^3$), this cubic surface must not be mixed up with the ring parabolic cyclid (fig.8, Serie X n°5 Schilling collection) of equation

$$Z(X^2 + Y^2 + Z^2) + 4(X^2 - Y^2) - 16Z = 0.$$

As a matter of fact, the latter has two double lines. Both of them separates the space into two isometric subsets which are swapped to each other by the half-turn rotation about the axis

$$X = Y \text{ and } Z = 0.$$

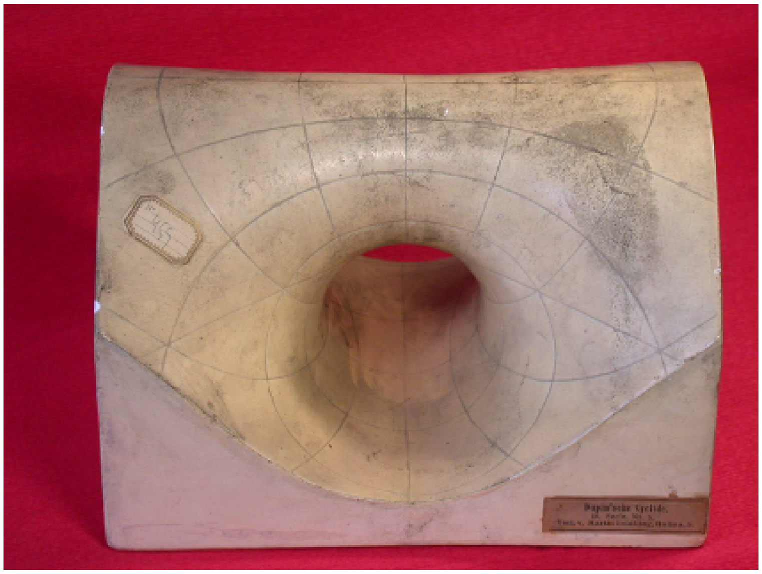

Figure 8. Ring parabolic cyclid

6. Dandelin model

The educational aim is clear (fig.9).

On a wooden cone divided into two parts in order to make two inscribed wooden spheres visible, one notices, on one hand, a brace rod figuring a generating line of the cone, and, on the other hand, a metallic ellipse whose plane is tangent to both wooden spheres. The points of tangency on the spheres materialize the focus of the ellipse (first part of Dandelin theorem), whereas the touching circles of the spheres with the cone define planes meeting that of the ellipse along its directrices (second part of Dandelin theorem).

7. A model of descriptive geometry

This model built by Joseph Caron in July 1914, was obviously intended for illustrating a lecture on descriptive geometry (fig.10).

The purpose of this field, today missing, and created by Gaspard Monge was to represent a 3-dimensional object by two orthogonal projections on two orthogonal planes. The object we are dealing with is the intersection of two ruled surfaces. The first one is a right cylinder the base of which is a

Figure 9. Dandelin model

Figure 10. Intersection of two surfaces

Figure 11. Sievert surface

logarithmic spiral and its generating lines are materialized by a green sewing thread. We get the second one by dilating the spirals and corkscrewing the cylinder, in the same way one produces a one-sheeted hyperboloid from a revolution cylinder, that is by connecting (with a red sewing thread) the generic point of the upper spiral, with the point of the lower spiral the polar angle of which being shifted by a constant. The intersection curve of the two surfaces occurs as a sequence of small pearls.

8. Sievert surface

Here, we are speaking about a plaster model of the Schilling collection designed by Georg Heinrich Sievert in 1886 in view to produce an explicit example of Joachimstal surface parametrized by elementary functions and which is not of revolution. One distinguishes three bulbs and one guesses a fourth one hidden behind. They are engraved with a network of lines of curvature (fig.11).

Figure 12. Etruscean Venus

Some are plane, and the others are spherical. That is the reason of the presence of smooth round spheres which figure four spheres of geodesic curvature (two big and two small) meeting the surface orthogonally along curvature lines, and, so doing, enforcing a theorem due to d'Ossian Bonnet (see [3]).

9. Two recent models

The first one (fig.12) is a stable model of the Klein bottle with twelve pinch-points (see the section on the one-sided cyclid).

We get it as connected sum of two roman Steiner surfaces. It has been made by Stewart Dickson in 2005 on a laser prototyper. It has been named Etruscean Venus by George Francis [4].

In the same line of thought, the next picture(fig.13) shows a Boy surface of degree six generated by a family of ellipses and opened by windows in order to make visible the whole structure [1].

Figure 13. Boy surface of degree six

It is the same surface as on the right in figure 1. It has been made by Gregorio Franzoni in 2008 on another kind of laser prototyper (see the website www.mathshells.com).

Both models, not yet belonging to the IHP collection, show what on can imagine to realize in order to update it, thanks to the present-day technics.

10. The Gömböc

The most recent acquisition is the Gömböc, constructed in 2006 in order to prove the existence of a mono-monostatic body conjectured by V. Arnold. This object in aluminium has a single stable resting position as well as a single unstable one. In the two dimensional case, it is well known that a convex set lying on a line has at least four static equilibria. We do assume that there are no degenerate static equilibrium unlike in the disk where all are. The picture is different in the three dimensional case. As conjectured by Arnold there exist convex homogeneous bodies in $\mathbb{R}^3$ admitting only two equilibrium states. We must count stable, saddle-type as well as unstable equilibria. Amazingly, such an example has been found by an engineer and an applied mathematician, Gábor Domokos and Péter Várkonyi, starting

FIGURE 14. Gömböc numbered 1928 in reference to the foundation of the Institute Henri Poincaré

from physical models and gradually modelling until it works, and then proving that the result, the so-called Gömböc, is correct. This is a remarkable example of what concrete models can bring to mathematicians.

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