INVERTING BEAUTY

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Abstract

In this paper we give a simple application of spherical inversion, the most elementary among the non elementary geometric transformations, and of some of its generalizations.

The principal motivation was an attempt to increase the interest for mathematics in high school students by proposing an easy but mathematically rigorous technique for creating new images, new shapes and, by means of 3D printing, new nice material objects. Also in order to put once again in evidence the possibility that mathematics can have something in common with Nature and the Arts.

Amongst the generalizations of inversion (see [Bl], [Hi], [Ep], [S1]), we find ideally more close to our point of view the hyperbolic inversion due to G. V. Schiaparelli[1], an important Italian astronomer not as much known as a geometer, who in [S2], in 1898, tried to represent organic forms and the change from one species to another through geometry (see, e.g., [Gi-Gu]).

1 Introduction

Nature offers our eyes every day extraordinarily beautiful forms, that look always the same and always new, but that never fail to amaze us. An example is represented by flowers: the pleasure we get from them is one of the most intense. Maybe is this one of the reasons why artists often give in to the temptation to reproduce them, sometimes emulating, sometimes interpreting nature, that is, deforming their shape.

Everyone has their own preferences. Many have predilection for roses, but some find calla-lilies (or arum lilies) more fascinating, because of their slender elegance, which firmly soars spiraling upward. It's not hard to find callas in vases and gardens, or in paintings. A bouquet of callas, as that in Fig.1, is an

[1]Giovanni Virginio Schiaparelli (1835-1910), astronomer and historian of science, senator of Kingdom of Italy, Bruce and Royal Astronomical Society gold medalist, discovered groups of straight lines (*canals*) on Mars, raising doubts on existence of life on that planet, and gave an explication of shooting stars as residues of comets. The relationship between his hyperbolic transformation and standard inversion was observed by Luigi Cremona.

Fig. 1 Bouquet of calla-lilies

interesting subject to several painters, who may have been attracted by their geometric profile and their nearly evanescence.

Drawings and paintings with callas as subject can be easily found on the net, taken from museums and from more or less important art galleries.

Fig. 2 (a) Calla-lilies (b) Great Peacock Moth

In Fig.2a and Fig.2b we have reproduced a watercolour painting by Stanis Dessy, a Sardinian artist (1900-1986), and a Van Gogh's picture.

It is interesting to know that also in mathematics one can find a *calla*: the extremely elegant and beautiful surface which bears her discoverer's name, Ulisse Dini, (see [Di]). It can be drawn with *Mathematica* (see for example [Ca-Gr]) by using the parametrization given by the map

$$X(u,v) = \left\{ b \sin u \sin v, b \sin u \cos v, b \left(\cos u + \ln\left[\tan\left(\frac{u}{2} \right) \right] \right) + cv \right\}. \quad (10.1)$$

For $a = 32$, $b = 5$, $u \in [-3\pi, -0.5]$ and $v \in [0.02, \pi/2 - 0.02]$, we get

Fig. 3 Dini's Surface

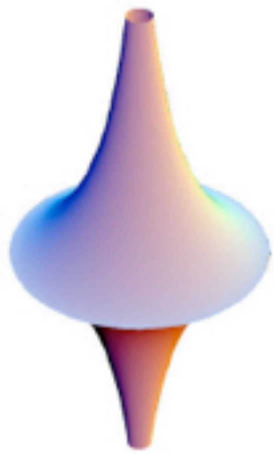


Fig. 4 Pseudosphere

Dini's surface can be obtained through an isometric deformation from the Eugenio Beltrami's *Pseudosphere* [2], which is parametrized by

$$X(u,v) = \left\{ \sin u \sin v, \sin u \cos v, \cos u + \ln\left[\tan\left(\frac{u}{2} \right) \right] \right\}.$$

Therefore these two surfaces, even though it does not appear evident at first glance, share the property of being curved in the same way in every point. More precisely, they have Gauss curvature K constant and equal to -1.

One may wonder whether it is possible to create by means of mathematical tools a composition resembling any of those in Fig.1 and in Fig.2. In order to draw a bouquet from the Dini's surface, it is necessary to find a way of *bending* it in a proper way, to obtain a visually pleasing composition.

It is important to observe that this can be done through a *spherical inversion*. This is a non elementary (i.e. non linear) geometric transformation, in fact one of the simplest, besides rigid motions (the congruences) of Euclidean geometry.

In the following sections we will briefly recall the notion of inversion, some of its well known properties and some generalizations, not as much well known. But first of all we show what our bunch of "flowers" looks like:

[2]Section 2 of Livia Giacardi's interesting paper [Gi] of 2013 ESMA proceedings is devoted to the Beltrami's cardboard model of this pseudo spherical surface.

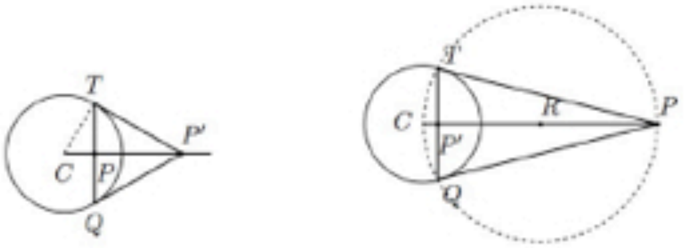
The number r^2 is the *power* of inversion.

From the definition it is easy to determine the mutual position of P and P' with respect to γ:

1. $P = P'$ if and only if P lies on the circle γ.
2. If P is inside γ, then P' is outside γ, and P' is inside γ if P is outside.
3. $(P')' = P$ (that is, the inverse of the inverse of P is P).

Moreover, there are very simple geometric constructions to find the inverse P' of $P \neq C$. Besides the trivial case, when P belongs to γ, we have two cases:

1. **The point P is inside γ.** Let TQ be the chord of γ through P perpendicular to $\overrightarrow{CP}$. Then the inverse P' of P is the point of intersection of the tangents to γ at T and Q.
2. **The point P is outside γ.** Let R be the midpoint of the segment CP, and σ the circle with center R and radius $\overline{CR} = \overline{PR}$. Then σ intersects γ in T and Q, PT and PQ are tangent to γ, and the inverse P' of P is the intersection of TQ and CP.

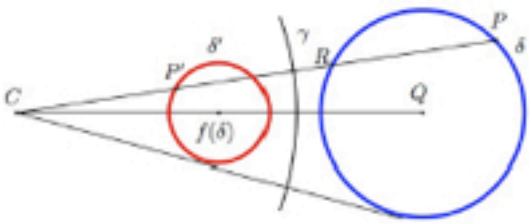


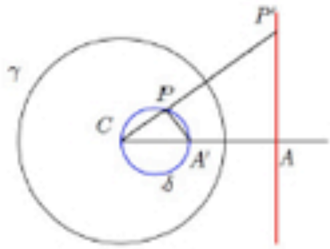
Once we have learnt how to find the inverse of a given point, it is interesting to see how are the inverses of sets of points. If the figure to invert is a circle, the result, simple and surprising, is a Steiner's theorem that in [Pa], p.178, is called the **fundamental theorem of inversion**:

Theorem 2.1 *The inverse of a circle is a circle.*

More precisely one has the following two cases:

1. Let γ be a circle of radius r and center C, δ a circle of radius s and center Q. Assume C outside δ and let k be the power of C with respect to δ. Let f the dilation with center C and ratio $\lambda = r^2/k$. Then the image δ' of δ under inversion in γ is the circle of radius $\lambda \cdot s$ whose center is the image $f(Q)$ of Q.
2. Let δ be a circle passing through the center C of a circle γ. The image of δ minus C under inversion in γ is a line ℓ not through of the center C; the line ℓ is parallel to the tangent to δ at C.





It is also well worth considering inversions of other conic sections. We must here remark that, to represent with the program *Mathematica* the inverse of a parametrized curve, it is convenient to translate condition (10.2) in the formulas relating the coordinates of a point P and those of its inverse. These formulas are obtained by observing that, when the point P describes the curve α, the inverse curve α', is drawn by the point P' given by

$$P' = C + \frac{r^2(P-C)}{||P-C||^2}. \tag{10.3}$$

Here we have denoted by $||P-C||$ the length of the segment $\overline{CP}$.

Let us show (in red) some inverse curves of conics. Inversion of a parabola with respect to circles centered at its vertex and its focus gives, respectively, a *cissoid of Diocles* and a *cardioid*:

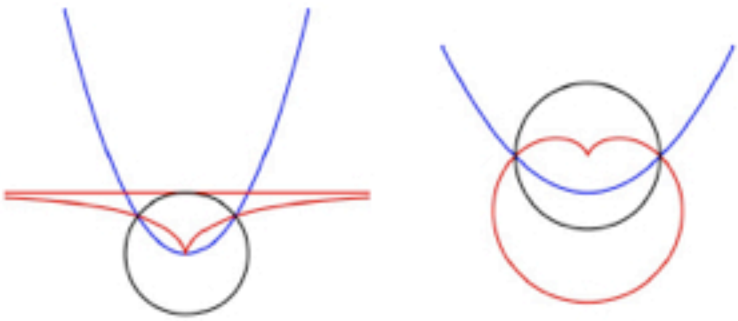


Fig. 6 A parabola with its inverse with respect to circles centered in its vertex (*left*) and in its focus (*right*)

For an ellipse, taking the circle of inversion centered in a vertex, in the center and in a focus of the ellipse, we obtain respectively a *witch of Agnesi*, a *lemniscate of Booth* and a *limaçon of Pascal*

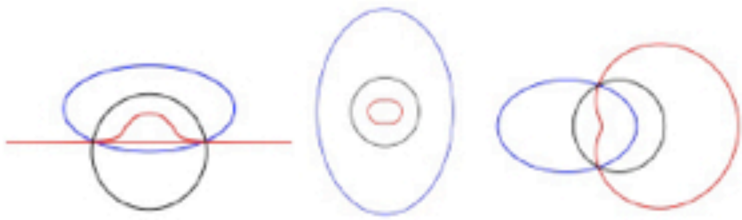


Fig. 7 An ellipse with its inverses with respect to circles centered in one of its vertices (*left*), in the center (*middle*) and in one of its foci (*right*)

When the conic we want to invert is a hyperbola and the circles of inversion are chosen as for the above ellipse, we get a *strophoid*, a *lemniscate of Bernoulli* and a *limaçon of Pascal*, respectively.

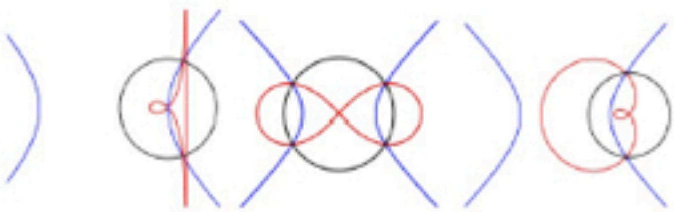


Fig. 8 A hyperbola with its inverses with respect to circles centered in one of its vertices (*left*), in its center (*middle*) and in one of its foci (*right*)

But it is also interesting to see in what way the inversion deforms triangles and squares.

Fig. 9 Inverses of a triangle contained in (*left*) and containing (*right*) the circle of inversion

Fig. 10 Inverses of a square with respect to a circle inside the square, and to a circle outside the square

3 Inversions in three dimensions

Let us now consider the inversion with respect to a sphere, to explain the way Fig.5 was obtained.

Conditions (10.2) and (10.3) do not change when we want to invert a point $P \neq O$ in ordinary space with respect to a sphere centered in O and of radius r. But now, besides the curves, we can invert planes, spheres, quadrics and more complicated surfaces, as it can be seen in [Ca-Gr].

For example, in Fig.11 are represented the Möbius strip

$$X(u,v) = \left\{\cos u + v\cos\frac{u}{2}\cos u, \sin u + v\cos\frac{u}{2}\sin u, v\sin\frac{u}{2}\right\}$$

and its inverse with respect to the sphere of radius 2 centered in the origin:

Fig. 11 A Möbius strip with its inverse

In Fig.12 we can see a torus and, on the right, one of the famous Dupin cyclides, which held also J. C. Maxwell's interest (see [Mx1]). This cyclide can be obtained by inverting the torus

$$X(u,v) = \{\cos u(8+3\cos v), \sin u(8+3\cos v), 3\sin v\}$$

with respect to the sphere centered in the point $C = (0,2,0)$ and of radius 2.

Now we draw in Fig.13 two of the surfaces we need to compose the image in Fig.5.

Fig. 12 A torus with its inverse

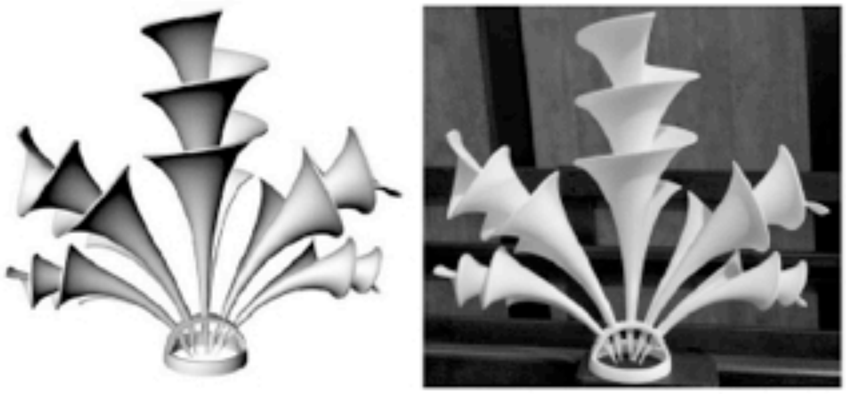
The inverses of the Dini's surface given by parametrization (10.1) with respect to the sphere centered in $(-0.63, 150, -162)$ and of radius 150 and with respect to the sphere centered in $(-0.63, 280, -162)$ and of radius 280 are:

Fig. 13

And finally we are able to obtain the bouquet in Fig.5. We just put together the surfaces in Figures 3 and 13 and the two other inverses that we get by inverting the Dini's surface with respect to the sphere centered in $(-0.63, -150, -162)$ of radius 150, and to the sphere centered in $(-0.63, -280, -162)$ of radius 280.

4 Digression. 3*D* Printing of the Dini bouquet

Probably the reader knows what 3D printing is. During the last 4 - 5 years it sticked out from technical reviews and specialist environment to come to mass reviews and TV. In a certain sense, 3D printing "brings to real life" solid objects who live in the virtual worlds created by computers. Thanks to it we can realize and keep in our hand a very precise copy of the Dini surface bouquet. Those who do not know enough about 3D printing techniques will find some details further on this paragraph. The picture below shows a virtual model of the bouquet (on the left) and a photography of the corresponding physical model of it.



Let us spend some words more about this technique. First of all, any application of these or other surfaces in physical world bases on the realization of material models of such objects. By means of scientific, design or engineering software it is possible to get 3D models, i.e. models that exist in a virtual three dimensional space in the memory of a computer and of which we can see in perspective some projections and animations giving us the impression of being watching real objects.

Techniques of *rapid prototyping* and *3D printing*, born in the end of the eighities of last century, allow to carry out the next step, that is to take mathematic surfaces to the concrete, real world in the form of tangible objects, which we can hold in our hands, rotate, and observe from different angles, to get a precise idea of their geometrical and topological properties. The mentioned terms indicate a series of techniques widely used to build conceptual and functional prototypes in several industrial fields, like automotive, electric household appliances, toys, jewels and medical fields, with not negligible applications in artistic, cultural and archeological areas. The main innovation introduced by these techniques (in the following we will use just the *3D printing* term, which generally indicates medium to low cost systems that do not require a specific technical competence to users, while the *rapid prototyping* term is associated to the industrial/professional systems; the bases of the operating principles are the same) is that they make realizable every kind of shape, no matter how complicated they are, with the sole condition that they represent real solid objects, i.e. not impossible figures. Shapes can be complex, can have back drafts, undercuts, inner canals of cavities, features that make them impossible to be realized by means of more traditional techniques such as lathe or CNC cutter (for example: a car's engine block, including the duct for the liquid coolant, or the accurate reproduction of an human skull). This is made possible thanks to the working method of *3D printers*. They decompose the 3D model to be realized –

usually a closed triangle mesh – into a collection of plane parallel sections. Then they execute the realization of every layer, in thickness variable between 0.05 and 0.5 mm depending on technology, until they produce a concrete part that corresponds to the virtual object. Materials used vary depending on technology: photopolymers, ABS resin, nylon, plaster, ceramic, metal.

Fig. 14 Photos of 3D prints of models of surfaces; From left: part of the Klein bottle in Lawson's version; Moebius band with circular boundary; Boy surface according to F. Apéry's parametrization (see [Ap]).

So, what do we have to do if we have the equations of a surface and we want a physical model of it? Unfortunately, the operation is not straightforward. In fact, the input needed for an RP system is a watertight polyhedral mesh which represents a real object. In the main applications of 3D printing (design of new products in industrial field) the solid model is produced by *solid modeling* software, specifically conceived to give outputs ready to use with 3D printers, or by reverse engineering techniques. There is a survey of scientific software which allow the representation and visualization of surfaces starting from their parametrization (equation), like *Mathematica*, *Maple*, *MatLab*, *MathCad*, but such representations are not usable on a 3D printer. It is necessary to shift from the bidimensional exhibits needed in visual/graphic environments to volume-including shells suitable to 3D printing environment. This idea can be transferred into a series of mathematical operational steps, whereof starting with a surface parametrization we end up having a closed mesh directly usable by a 3D printer, which effectively represents our surface. It is relatively simple to obtain a printable solid model for regular surfaces without multiple loci by using the basic tools of differential geometry: it suffices to define some parallel and normal surfaces to the given one to construct a solid shell around it. Otherwise, in presence of self-intersections and/or singularities (and these are often the most interesting cases) we need to solve nontrivial problems which involve more differential geometry and computational geometry and which request skills transversal to both the mentioned scientific fields.

We finish this short journey into the world of $3D$ printing saying that in the last decade 3D printers performances have been constantly increasing while their price has decreased. In a close future, that probably has already begun, it will be normal to have a $3D$ printer connected to our PC just like today everyone has a inkjet or laser printer. Today (2013) it is already possible to buy a small $3D$ printer kit, with basic performances, at a price as low as 600 dollars.

5 Notes on circular inversion

Historically, the interest for transformations of the entire plane and for the properties that are invariant with respect to them, to solve geometrical problems, comes from the development of projective geometry of the XIX century, mainly due to Gaspard Monge (1746-1818) and to Jean-Victor Poncelet (1788-1867).

As regards the inversion, to our knowledge the most authoritative references are, in chronological order, the F. Bützberger pamphlet [Bü], reviewed by Arnold Emch in the Bulletin of the American Mathematical Society, vol. 20 (1914), pp. 412 - 415, and the very interesting Boyd C. Patterson's paper [Pa] on the origins of the geometric principle of inversion. Another useful reference for circular inversion and its generalizations, including those mentioned in the last two sections of this paper, is [Ca].

Let us follow the more important stages of this route from the beginning, in chronological order.

The first paper containing *in nuce* the idea of inversion appears in 1600, when François Viète gives a solution of the tenth Apollonius problem [3]. On pages from 5 to 9 of [Vi], Viète presents a solution by using the center of similitude of two circles.

After more than two centuries, the belgian mathematicians (and good friends) Germinal Pierre Dandelin [Da] and Adolphe Quételet [Qu], also known for the so-called *Belgian Theorems* (see, e.g., [Hu]), arrive, independently one from the other, to the principle of inversion when studying the properties of the *focale* of conic sections (see, e.g., [Pa], p. 156) by means of stereographic projection. In particular, Quételet deduces the relation $rr' = R^2$ between *radii vectores* that are reciprocal with respect to a circle of radius R, and also the explicit analytic formulas of the circular inversion.

But the first who gives the precise definition of inversion and establishes and applies inversion is Jakob Steiner, during his researches on the geometry of

[3]PROBLEMA X: *Datis tribus circulis, describere quartum circulum quem illi contingant. (To draw a circle that touches three given circles in a plane.)*

the circle and the sphere through the theory of similar figures (see [Bü]). Later, in 1847, Joseph Liouville will give to the inversion the name of *transformation par rayons vecteurs réciproques*. Again from [Bü] it comes out that Steiner was interested in this study in the attempt to solve the following problem: given in a plane three circles, to find the locus of a point whose polars with respect to three circles pass through a point.

In 1831, Julius Plücker (see [Pl]) finds inversion within the theory of poles and polars (in the next section we shall recall the construction of poles and polars of a conic), while in 1832 Ludwig Immanuel Magnus, in [Ma], discovers inversion as a particular case of a bijective map between two planes.

In 1836, Giusto Bellavitis, informed, even if partially, about the results obtained by Dandelin and Quételet, realizes that inversion is a very useful instrument that, combined with other geometric tools, could lead to new theorems. His memoir [Bl] contains an elementary and very complete exposition of various geometric transformations such as those of similarity, projection, inversion, reciprocal polars, homology.

In 1842 there are two more simultaneous and independent discoveries of a *new principle* that is nothing but the inversion. The mathematicians involved are John William Stubbs and John Kells Ingram, both at Trinity College in Dublin; three of their papers on the subject even appear in the same volume of the *Transactions of the Dublin Philosophical Society* (see [St1], [St2], [In1] and [In1]).

In 1845, William Thomson (Lord Kelvin) communicates to Liouville his "principle of reciprocal points", a method he found useful to solve a certain problem in electricity. Louville develops the analytic theory of the Thomson's transformation that in this occasion he calls *transformation by reciprocal radii*.

The last we have to mention is August Ferdinand Möbius, that in 1855 undertakes in [Mö] a systematic study of inversion.

6 Generalizations

A quite reasonable question is whether there exist other similar, simple, geometric, useful constructions, that can also give rise to new shapes. In 1838, Giusto Bellavitis of Padua proposed in [Bl] a very natural generalization of the circular inversion by taking any conic instead of a fixed circle, and allowing the center of the inversion to be placed anywhere, and not only in a special position as, for example, in a center of symmetry.

Such a generalization can be obtained by considering that, as for the circle, for any conic there is a canonic, geometric way to find the point P' inverse of a point P. This happens thanks to the fact that a conic determines in its plane a correspondence between points P, the *poles*, and straight lines p, their *polars*.

For example, given an ellipse σ, if P is a point outside σ, its polar p is the straight line through the points Q and T where the two tangents to σ from P touch the conic.

When P is a point of σ, there is only one tangent to σ at P, and therefore this tangent coincides with the polar p of P.

If P is inside σ, any two straight lines $r_1, r_2, r_1 \neq r_2$, passing through P, intersect σ in four points A_1, B_1, A_2, B_2. Let Q and T be the intersections of the lines a_1, b_1 and a_2, b_2, respectively tangent to σ at the points A_1, B_1 and A_2, B_2. Then the polar p of P is the straight line through Q and T. The following figures illustrate the first and the third case.

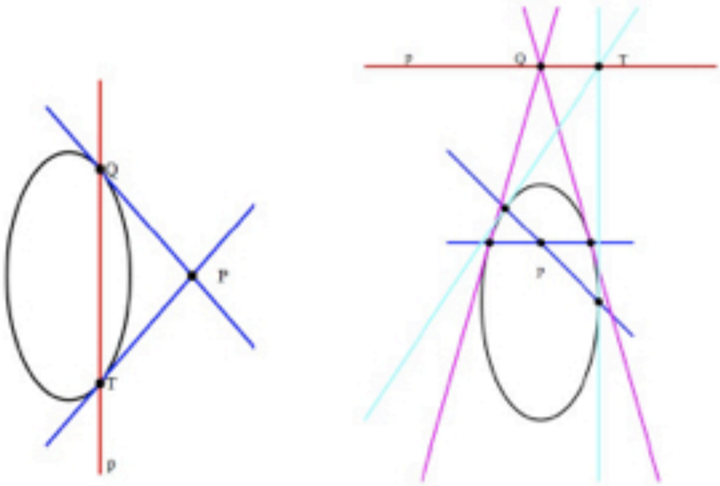


Fig. 15 Polars (red) of a point P outside or inside an ellipse

Therefore, according to Ludwig Immanuel Magnus and Julius Plücker (see [Ma] and [Pl]), we can define the inversion with respect to a conic σ in the following way.

Definition 6.1 *Let A be a point fixed as origin in the plane of σ. Then the inverse P' of a point P of this plane is the intersection between the polar of P with respect to σ and the straight line through P and the origin A.*

In Fig.16 are represented an ellipse and the construction of P' when the point P is outside (left) and inside (right) an ellipse:

If P belongs to σ, its polar line is the tangent to σ at P and the straight line through P and A intersects this tangent at P. Thus $P = P'$.

Except for the origin A and the two points (real or imaginary) where the tangents from A to σ touch σ (these are the three *principal points*), every point P has only one inverse P'.

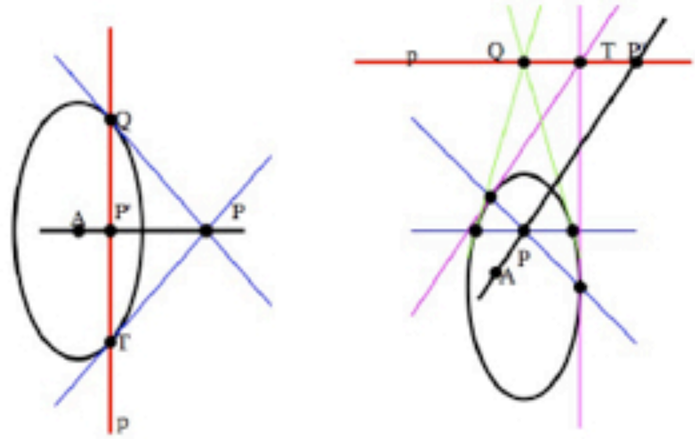


Fig. 16

It is not difficult to foresee that the study of the inversions with respect to the different conics and to various origins can be even rather complicated. This work has been achieved in 1865 by T. A. Hirst in the paper *On the quadric inversion of plane curves* [Hi][4], where one can find an ample and detailed essay on the subject. Here we shall only mention and illustrate by drawings the more interesting and useful of the special cases corresponding to particular choices of the conic and of the origin. In the following, the *fundamental conic* is the conic with respect to which we invert points.

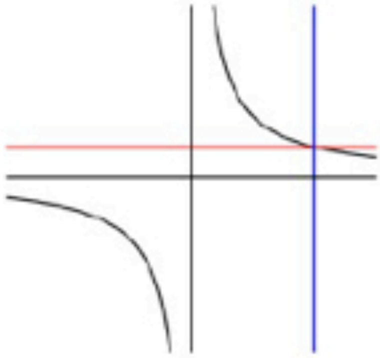
(I) **The fundamental conic is a hyperbola with its centre at the origin A.**

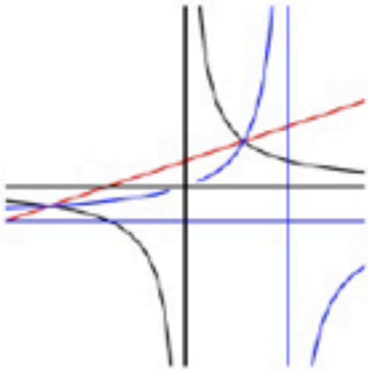
The inverse of every straight line parallel to one of its asymptotes is a straight line parallel to the other asymptote, and the two straight lines intersect on the fundamental hyperbola.

The inverse of every other straight line is a hyperbola passing through the origin, and having its asymptotes parallel to those of the fundamental hyperbola.

The inverse of any hyperbola which does not pass through the origin, but has its asymptotes parallel to those of the fundamental conic, is an hyperbola possessing the same properties; and if the hyperbola we want

[4]An Italian version of this paper, published in *Annali di Matematica Pura e Applicata*, Serie 1, Dicembre 1865, vol. 7,1, pp. 49-65, with title "Sulla Inversione quadrica delle curve piane", is due to Luigi Cremona. Here is how the paper is introduced: *We consider a good and useful thing to bring this important and very elegant work of our friend, Mr. Hirst, to the knowledge of the readers of the Annali.* (*Stimiamo cosa buona e utile il far conoscere ai lettori degli Annali questo importante ed elegantissimo lavoro del nostro amico, il Sig. Hirst.*) (Luigi Cremona)





to invert has centre at the origin, its inverse will have centre in the origin as well.

(Ia) The conic is an equilateral hyperbola.

Choosing again the origin at the centre of the hyperbola, the method of inversion becomes identical with the *hyperbolic transformation* investigated by Giovanni Virginio Schiaparelli, in his interesting memoir *Sulla trasformazione geometrica delle figure* (see [S1]).

(II) The fundamental conic is an ellipse and the origin is at its centre.

The inverse of every straight line in the plane will be an ellipse passing through the origin, and at the same time similar, as well as similarly placed to the fundamental ellipse (that is, in the two ellipses, the axes of symmetry corresponding in the similarity are parallel).

Every ellipse not passing through the origin, but similar and similarly placed to the fundamental one, has for its inverse an ellipse with the same properties; and should the primitive be likewise concentric with the fundamental ellipse, so also will be the inverse:

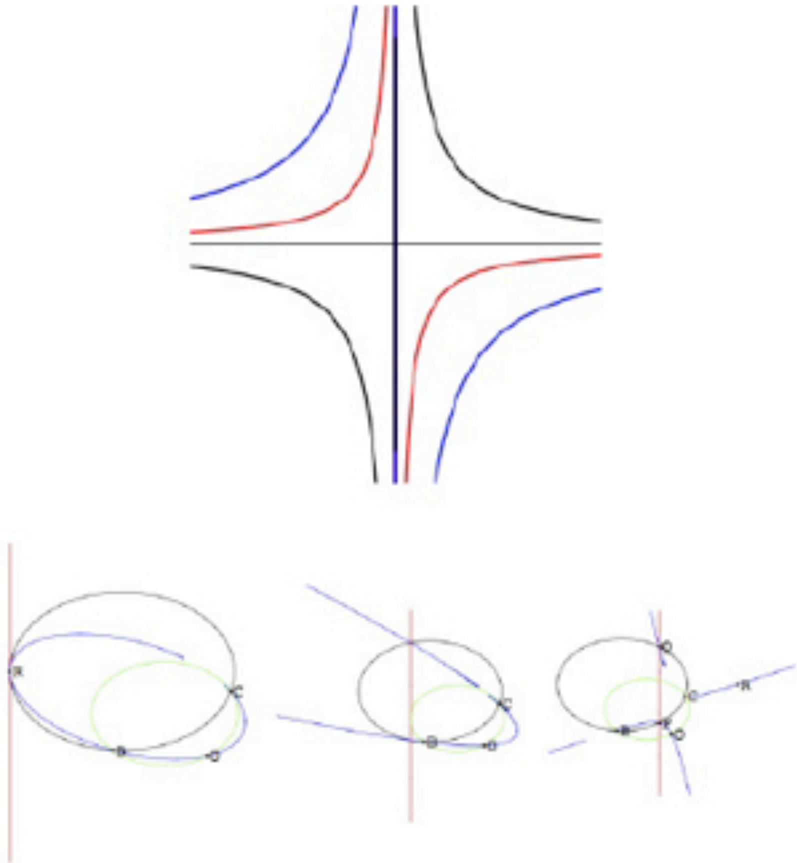


Fig. 17 Inversion of three parallel straight lines

7 Schiaparelli's hyperbolic inversion of some surfaces

In this section, we illustrate by some examples how the Schiaparelli hyperbolic inversion transforms figures and bends surfaces. Other examples can be found in [Pe].

To obtain a generalization of the spherical inversion, Schiaparelli substitutes the sphere with a quadric (see [S1]). Here we only consider the particular case when the quadric is a precise hyperboloid of two sheets.

- The circular cylinder of parametrical equations

$$\begin{cases} x(u,v) &= a\cos u \\ y(u,v) &= a\sin u \\ z(u,v) &= v \end{cases}$$

with respect to the hyperboloid of two sheets of equation $xy + xz + yz = 1$

has as inverse

Fig. 18 Hyperbolic inverse of a cylinder

- The inverse of the pseudosphere of parametric equations

$$\begin{cases} x(u,v) &= a\cos u \sin v \\ y(u,v) &= a\cos u \cos v \\ z(u,v) &= a\cos v + a\ln(\tan(\frac{v}{2})) \end{cases}$$

with respect to the same hyperboloid is

Fig. 19 Hyperbolic inverse of a pseudosphere

- Next we consider the Dini's surface of parametrical equations

$$\begin{cases} x(u,v) &= a\cos u \sin v \\ y(u,v) &= a\cos u \cos v \\ z(u,v) &= a\cos v + a\ln(\tan(\frac{v}{2})) + bu \end{cases}$$

Its inverse with respect to the same hyperboloid of two sheets is

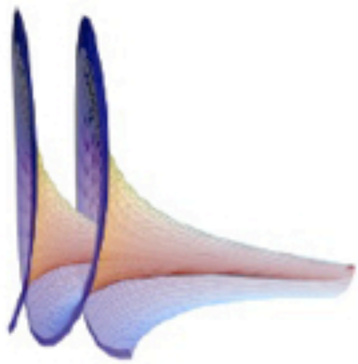


Fig. 20 Hyperbolic inverse of a Dini's surface

- Finally we apply the same inversion to the tori of parametric equations

  $X_1(u,v) = \{\cos u(8+3\cos v)-12, \sin u(8+3\cos v)-12, 3\sin v+20\}$

  and

  $X_2(u,v) = \{\cos u(8+3\cos v)-20, \sin u(8+3\cos v)-20, 3\sin v-20\}$.

  From the left to the right, the corresponding inverse surfaces are

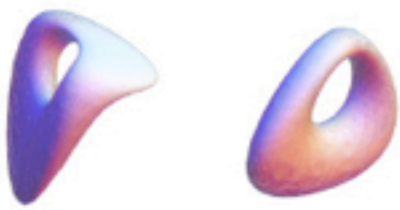


Fig. 21 Hyperbolic inverse of a Dini's surface

Remark A modern analytic treatment of inversion with respect to a general quadric of a n-dimensional vector space, endowed with a non-degenerated symmetric bilinear form, can be found in the 1983 D.B.A. Epstein's lecture notes [Ep].

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