M.C. Escher's Use of the Poincaré Models of Hyperbolic Geometry

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Abstract
The artist M.C. Escher was the first artist to create patterns in the hyperbolic plane. He used both the Poincaré disk model and the Poincaré half-plane model of hyperbolic geometry. We discuss some of the theory of hyperbolic patterns and show Escher-inspired designs in both of these models.

1. Introduction

The Dutch artist M.C. Escher was known for his geometric art and for repeating patterns in particular. Escher created a few designs that could be interpreted as patterns in hyperbolic geometry. Figure 1 is rendition of Escher's best known hyperbolic pattern, *Circle Limit III*. Escher created his hyperbolic patterns by hand,

Fig. 1. A computer rendition of the *Circle Limit III* pattern.

which was a very tedious and time consuming process, since the motifs were of different sizes and slightly different shapes. So about 30 years ago my students and I were inspired to create such patterns using a computer, which could transform the motifs almost instantly. In this paper we show some of the hyperbolic patterns we have generated.

We begin with a brief history of the creation of artistic hyperbolic patterns. Then we review the Poincaré models of hyperbolic geometry, and repeating patterns. With that background, we next show sample patterns from both the disk and half-plane models. Finally, we indicate possible directions of further research.

2. A Brief History of Hyperbolic Art

Euclidean, spherical (or elliptic), and hyperbolic geometry are sometimes called the "classical geometries". The Euclidean plane and the 2-dimensional sphere are familiar since they can be embedded in the 3-dimensional space in which we live. However, there is no smooth isometric embedding of the hyperbolic plane in Euclidean 3-space, as proved by David Hilbert more than 100 years ago [6]. Thus we must rely on non-isometric models of it. This is probably the reason for the late discovery of hyperbolic geometry by Bolyai, Lobachevsky, and Gauss almost 200 years ago. And it wasn't until the late 1860's that Eugenio Beltrami discovered what are now called the Poincar´e disk and half-plane models of the hyperbolic plane.

Almost a century later Escher received a copy of a paper from the Canadian mathematician H.S.M. Coxeter[1]. The paper contained the hyperbolic triangle pattern shown in Figure 2. Escher said that the Figure 2 pattern gave him "quite a shock" since it showed him how to make a repeating pattern with a circular limit (hence the name for his "Circle Limit" prints); he was already familiar with patterns with point limits (with dilation symmetries) and "line limits". Thus inspired, Escher created *Circle Limit I* in 1958, a rendition of which is shown in Figure 3. Over the next two years Escher created three more "Circle Limit" prints:

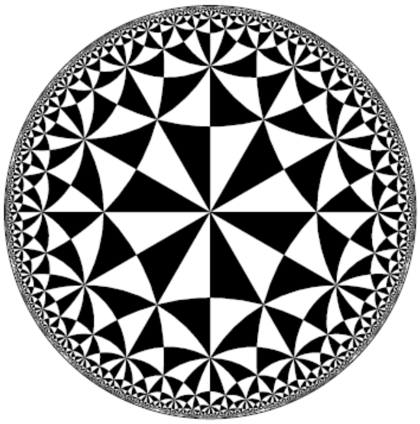

Fig. 2. The {6,4} tessellation

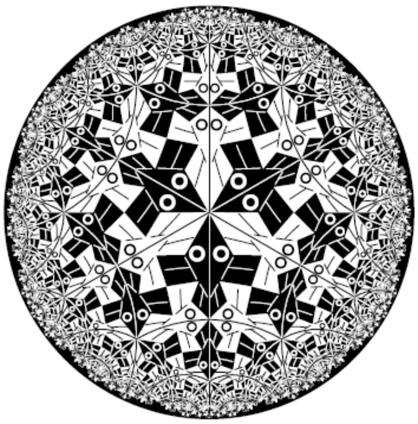

Fig. 3. A *Circle Limit I* rendition

Circle Limit II, *Circle Limit II* (shown in Figure 1 above), and *Circle Limit IV*. For more information, visit the official Escher web site [4]. Twenty years later, my students and I were in turn inspired to re-create Escher's four "Circle Limit" patterns using computer technology [2]. However the program we wrote was more general than required to reproduce Escher's "Circle Limit" patterns, so we created a number of new hyperbolic patterns. Another reference for the theory of computer generated hyperbolic patterns is [3].

3. Repeating Patterns and the Poincaré Disk and Half Plane Models

A model of hyperbolic geometry represents the basic elements of that geometry (points, lines) by Euclidean constructs. Conversely, as Beltrami showed, there are models of Euclidean geometry within hyperbolic geometry, so that that two geometries are equally consistent.

In the *Poincaré disk model* of hyperbolic geometry the hyperbolic points are represented by Euclidean points within a bounding circle. Hyperbolic lines are represented by (Euclidean) circular arcs orthogonal to the bounding circle (including diameters). The edges of the triangles in Figure 2 and the backbone lines of the fish in Figure 3, are hyperbolic lines. However the backbone lines of the fish in Figure 1 are not hyperbolic lines, but are so called equidistant curves (each point is the same distance from the hyperbolic line with the same endpoints on the bounding circle), which make an angle of about 80° with the bounding circle. The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model — the model is *conformal*, so that motifs retain the same approximate shape as they approach the bounding circle. This was a property of the disk model that appealed to Escher. Another desirable property was that an entire pattern could be displayed in a finite area, unlike "point limit" patterns which could theoretically grow outward to infinity and, "line limit" patterns which could also extend to infinity upward and to the left and right. However, equal hyperbolic distances correspond to ever-smaller Euclidean distances toward the edge of the disk, thus all the fish in Figure 1 are the same hyperbolic size, as are the triangles in Figure 2.

In the *Poincaré half-plane model* of hyperbolic geometry the hyperbolic points are represented by Euclidean points (x, y) in the upper half plane $y > 0$. Each hyperbolic line is represented by a (Euclidean) semicircular arc above the x-axis and with center on it (including vertical half-lines). Figures 4 and 5 show half-plane versions of Figures 2 and 3 respectively. The edges of the triangles in Figure 4 and the backbone lines of the

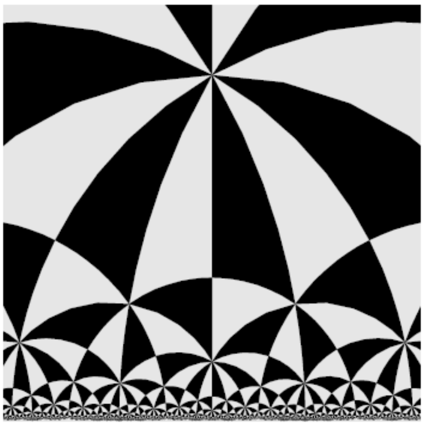

Fig. 4. A half-plane version of the triangle pattern of Figure 2

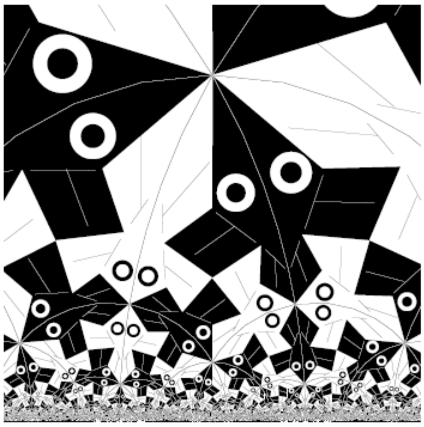

Fig. 5. A half-plane version of Escher's *Circle Limit I* pattern

fish in Figure 5 are all hyperbolic lines in this model. This model is also conformal, but was not as appealing to Escher as the disk model since it is unbounded. Still, Escher used this model to create two and possibly three patterns, which he called "line limit" patterns. The hyperbolic distance relationship is simple in this model —hyperbolic length is inversely proportional to the Euclidean distance to the x-axis.

A *repeating pattern* is a pattern made up of congruent copies of a basic subpattern or *motif*, where "congruence" is determined by the geometry in question. In Figure 1, the motif consists of one fish (disregarding color). In Figures 2 and 4, the motif can be either a black or a white triangle (again disregarding color). The motifs of Figures 3 and 4 consist of half a white fish together with an adjacent half of a black fish. It seems necessary to use repeating patterns to show the hyperbolic nature of the models. For instance, if there were just one triangle shown in Figures 2 or 4, we couldn't be sure if it was hyperbolic or just a curvilinear Euclidean triangle. For more information on hyperbolic geometry and its models, see [5].

4. Patterns in the Poincaré Disk Model

For completeness, we show renditions of Escher's patterns *Circle Limit II* and *Circle Limit IV* in Figures 6 and 7. Escher's last print, *Snakes* contains a pattern of interlocking "hyperbolic" rings near the circular boundary; the inner rings form a "point limit" (dilation) pattern. Figure 8 shows a complete pattern of the hyperbolic rings. Figure 9 shows a pattern like *Circle Limit III*, but with five fish meeting a right fin tips.

Fig. 6. A *Circle Limit II* rendition

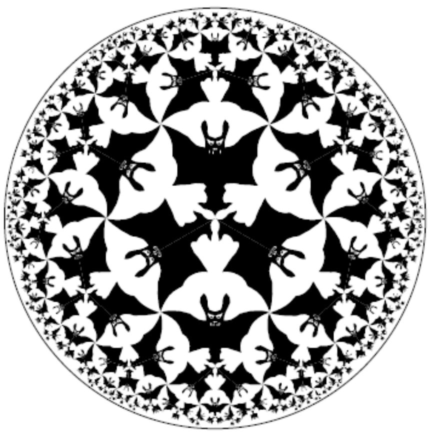

Fig. 7. A *Circle Limit IV* rendition

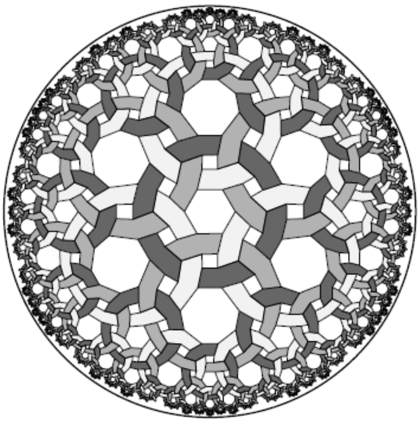

Fig. 8. A interlocking ring pattern inspired by Escher's *Snakes* print

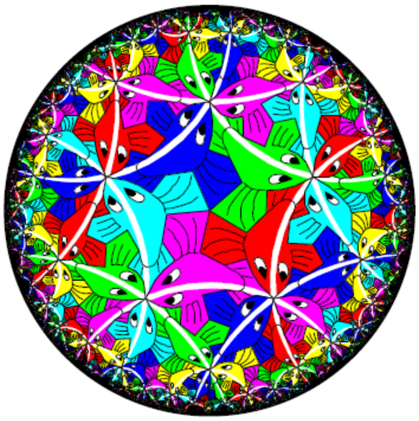

Fig. 9. A pattern of fish, five of which meet at right fins

5. Patterns in the Poincaré Half-plane Model

Escher seems to have created three "line limit" patterns. His *Regular Division of the Plane VI*, Figure 10, and *Square Limit* are based on the half-plane model, and *Regular Division Drawing 101* may be, but it is hard to tell since the lizards are modified in different ways. Figures 11, 12, and 13 show half-plane versions of *Circle Limit IV*, the pattern of Figure 9, and a fish pattern inspired by *Regular Division Drawing 20*.

Fig. 10. A Escher "line limit" pattern

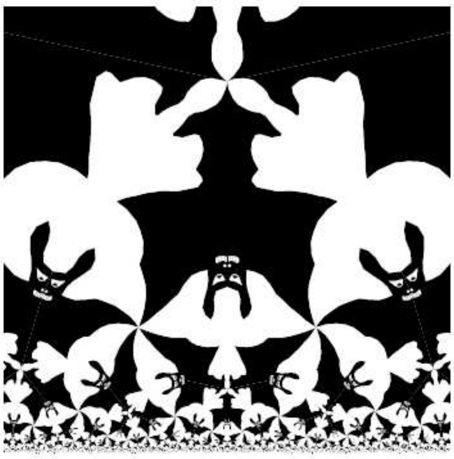

Fig. 11. A half-plane version of *Circle Limit IV*

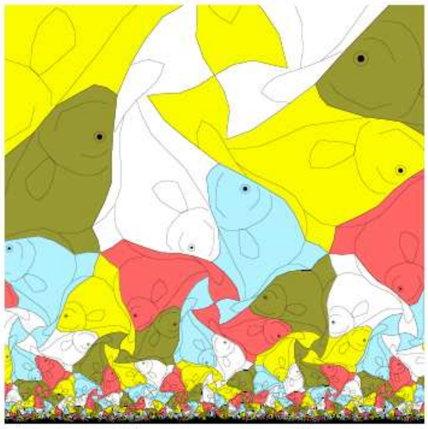

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Fig. 12. A half-plane version of Figure 9 **Fig. 13.** A half-plane fish pattern

7. Future Work

The disk model patterns we have created were designed using a drawing program that works in that model. This program has evolved over the years to have a number of useful features. However, the half-plane patterns that we have created were first designed using the disk model program and then transformed to the half-plane model. It would seem to be useful to have a program that would allow for the design of half-plane patterns using that model directly.

Also, we have just shown a few patterns in each of the models. It would be interesting to create many more such patterns.

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