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# The Hidden Symmetries of the Multiplication Table

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## Synopsis

In this article we explore some of the symmetries that hide in the distribution of numbers in the multiplication table of positive integers when viewed through modulo  $k$  arithmetic as we vary  $k$ .

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## Introduction

Let us start with the standard  $n$  by  $n$  multiplication table truncated to the first few positive integers.

0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10
0	2	4	6	8	10	12	14	16	18	20
0	3	6	9	12	15	18	21	24	27	30
0	4	8	12	16	20	24	28	32	36	40
0	5	10	15	20	25	30	35	40	45	50
0	6	12	18	24	30	36	42	48	54	60
0	7	14	21	28	35	42	49	56	63	70
0	8	16	24	32	40	48	56	64	72	80
0	9	18	27	36	45	54	63	72	81	90
0	10	20	30	40	50	60	70	80	90	100



In the following, we will color the squares of the multiplication table according to the values of the squares modulo  $k$  for various values of  $k$  and discover some aesthetically pleasing symmetries.

### The multiplication table modulo 2

We begin by assigning the color blue to every square in the multiplication table that is equal to 0 mod 2. Here  $k = 2$ . Then rows and columns numbered  $0, k, 2k, \dots$  will all be colored blue, leaving squares of side length  $k$  in the middle.

This will give a regular pattern with white squares of area

$$(k-1)^2 = (2-1)^2 = 1^2.$$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160

Note that we have extended the table a bit; indeed since the complete multiplication table on positive integers is infinite on two sides, we will continue to tweak the dimensions of the tables in what follows to display the emerging patterns more clearly.

### The multiplication table modulo 4

Now we will assign blue to every square in the multiplication table that is equal to  $0 \pmod 4$ . Since  $k = 4$  here, the repeating pattern will involve squares which themselves consist of  $3^2$  smaller squares. These squares have more structure than the mod 2 squares; the middle square in each such  $3 \times 3$  square will also be colored blue.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
0	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
0	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400

Next we try assigning the color blue to every square in the multiplication table that is equal to 0 mod 6. As  $k = 6$  here, this will give a pattern with  $5 \times 5$  white squares ( $((k - 1)^2 = (6 - 1)^2 = 5^2)$ ). These squares show a nice symmetric pattern; in particular they each include four blue squares, symmetrically distributed around their central square, which itself is white.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24				
0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48				
0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72				
0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96				
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120				
0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144				
0	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168				
0	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192				
0	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216				
0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240				
0	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264				
0	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288				
0	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312				
0	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336				
0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360				
0	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384				
0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408				
0	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	378	396	414	432				

A more interesting pattern can be discovered if we use multiple moduli, and corresponding to them, multiple colors. In the following figure, the numbers that are  $0 \bmod 2$  are colored red, and those that are  $0 \bmod 3$  are colored orange (with the orange taking precedence over the red in the case of multiples of 6).

[illegible]

Note that this time, our fundamental building blocks are  $6 \times 6$ , which makes sense, given that 6 is the least common multiple of our two moduli. The symmetry emerges from repeated copies of a  $5 \times 5$  square with a nice four-fold symmetry.

### Using three consecutive moduli and three colors

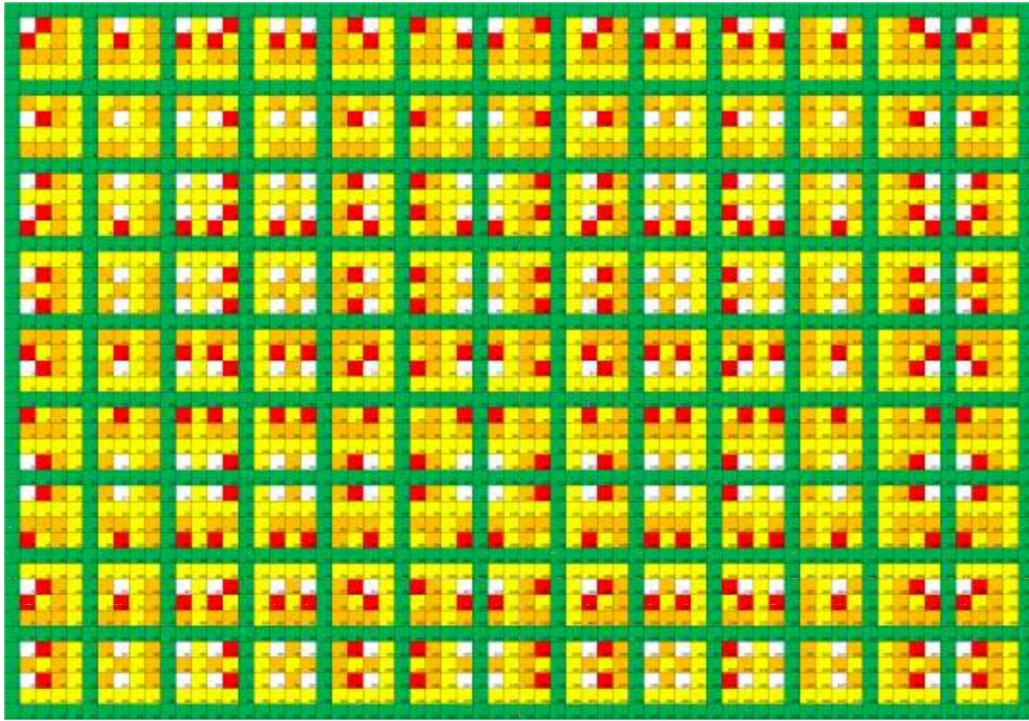
The next figure takes this a step farther, assigning red to the cells in the table that contain numbers that equal to  $0 \bmod 2$ , orange to the cells that contain numbers that equal to  $0 \bmod 3$ , and yellow to the cells that contain numbers that equal to  $0 \bmod 4$ .

[illegible]

This time, our fundamental building blocks are  $12 \times 12$ , which again makes sense, given that 12 is the least common multiple of our three moduli. The symmetry emerges from repeated copies of an  $11 \times 11$  square, which contains nine small  $3 \times 3$  squares which together create a nice four-fold symmetry.

### Using four consecutive moduli and four colors

Next we use four moduli and color those points that are  $0 \bmod 2$  to be red,  $\bmod 3$  to be orange,  $\bmod 4$  to be yellow, and finally  $\bmod 5$  to be green.

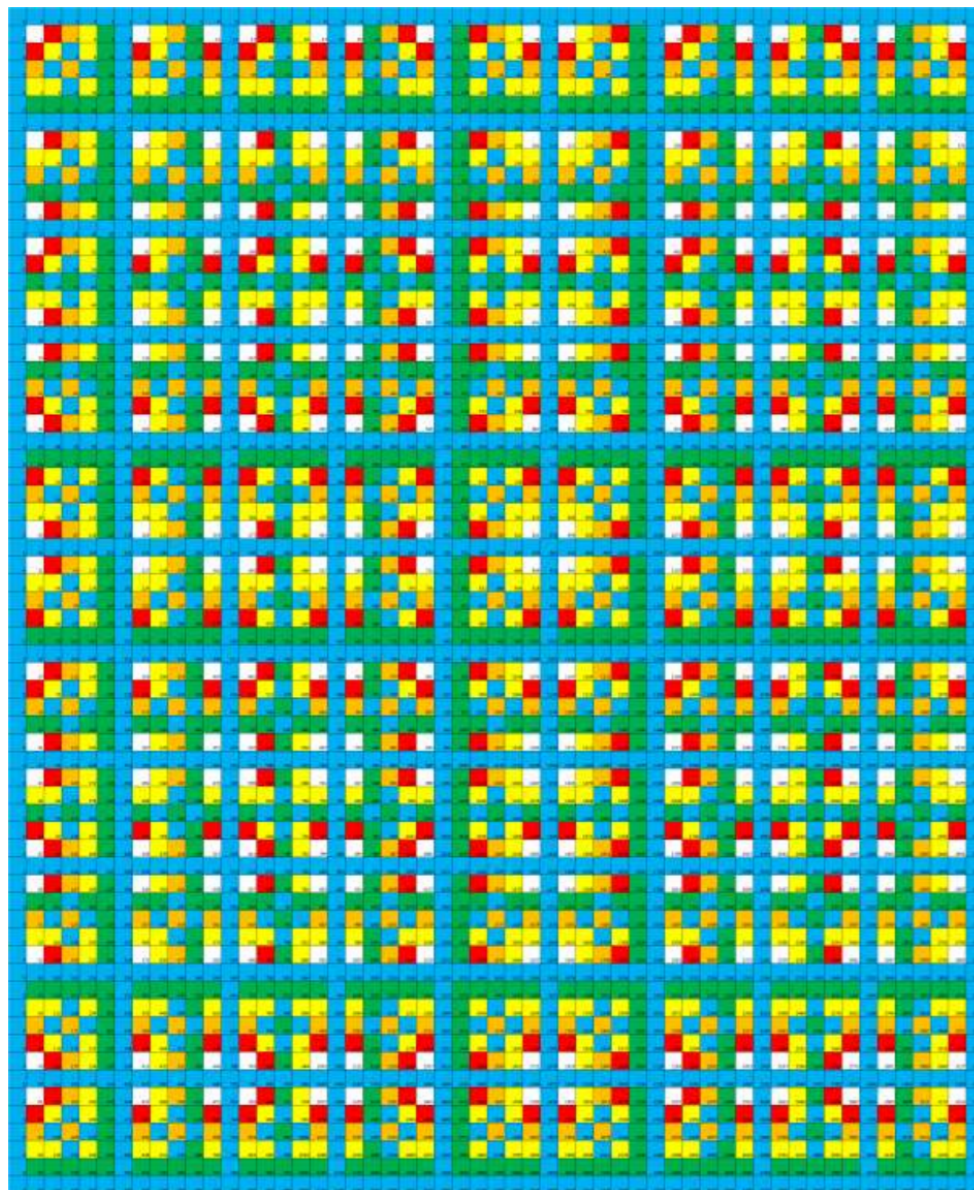


Note that this time, it is not obvious from simply looking at the figure what the repeating pattern should be. That is because the least common multiple of our moduli is 60, so to get a square building block, we need  $60 \times 60$  squares. And in our figure we went beyond 60 only in the horizontal direction. Still, even in this truncated form, the figure displays some aesthetically appealing features; the eye detects sections that are symmetric and repeated as well as a few axes of symmetry, and all in all, the distribution of the colors is quite pleasing.



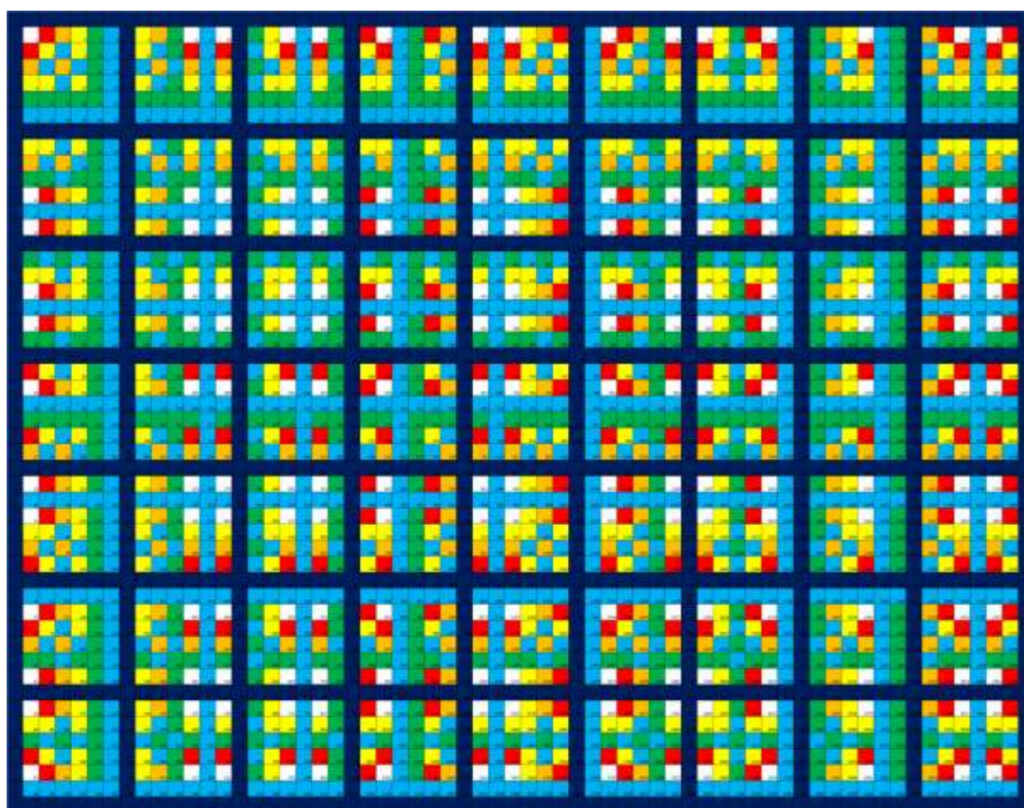
**Using five consecutive moduli and five colors**

The next figure uses red for mod 2, orange for mod 3, yellow for mod 4, green for mod 5, and blue for mod 6. Can you determine the dimensions of the building blocks of symmetry?



### Using six consecutive moduli and six colors

The next figure assigns the color red to the numbers that equal 0 modulo 2, the color orange to the numbers that equal 0 modulo 3, the color yellow to the numbers that equal 0 modulo 4, the color green to the numbers that equal 0 modulo 5, the color blue to the numbers that equal 0 modulo 6, and the color indigo to the numbers that equal 0 modulo 7.

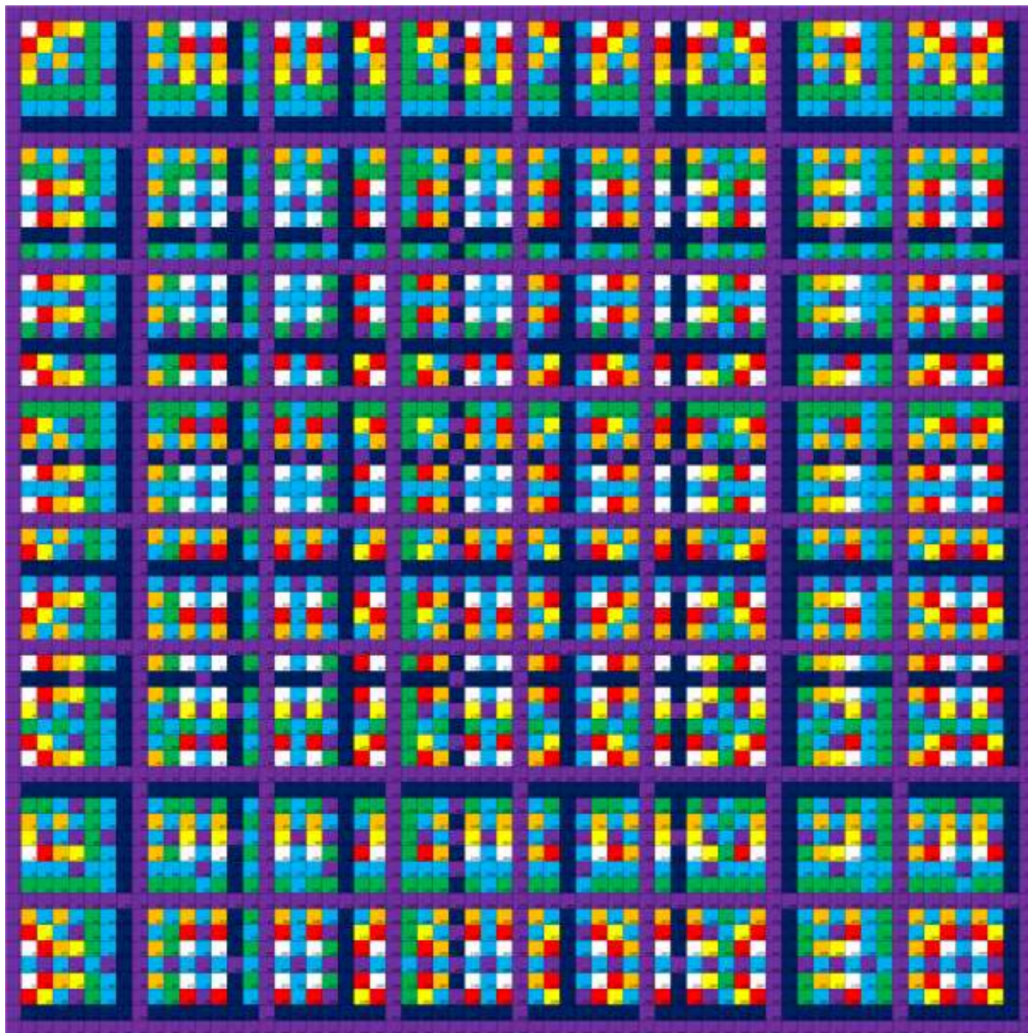


What patterns can you discern? Can you find any axes of (reflectional) symmetry? What should be the size of the fundamental (repeating) building blocks of symmetry in this case?



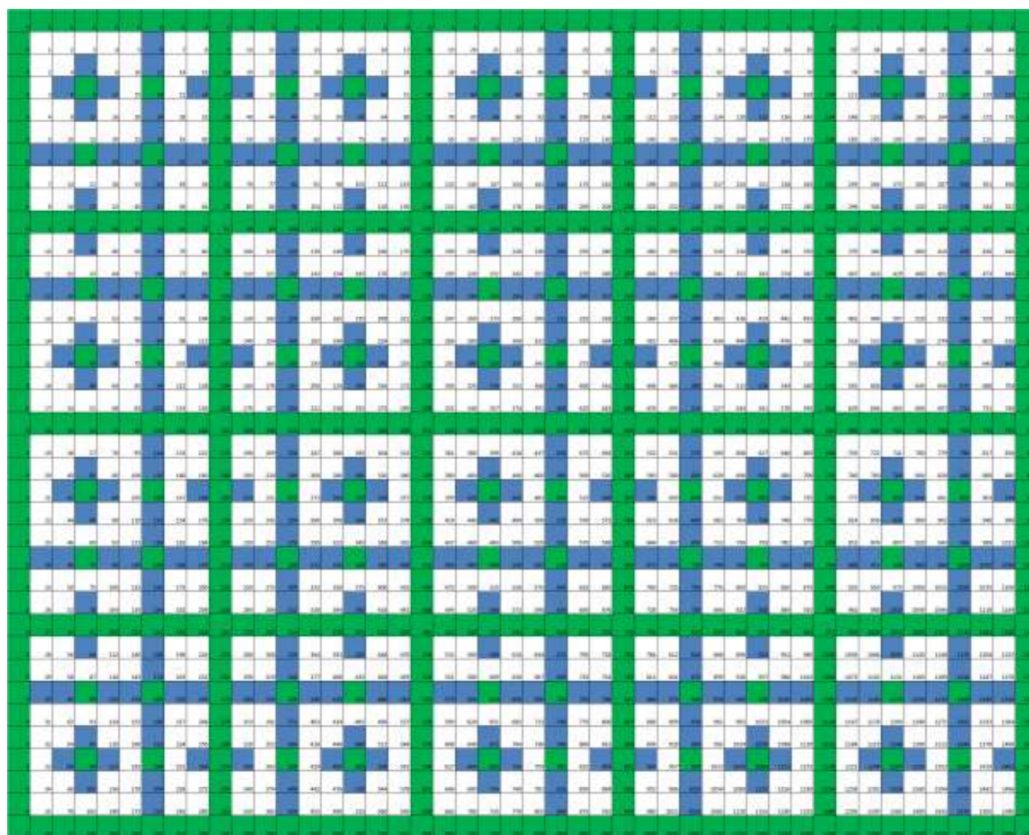
**Using seven consecutive moduli and seven colors**

Next we use the color red for numbers that equal to 0 modulo 2, the color orange for numbers that equal to 0 modulo 3, the color yellow for numbers that equal to 0 modulo 4, the color green for numbers that equal to 0 modulo 5, the color blue for numbers that equal to 0 modulo 6, the color indigo for numbers that equal to 0 modulo 7, and the color purple for numbers that equal to 0 modulo 8. Can you discern any patterns in the resulting figure?



### Using nonconsecutive moduli

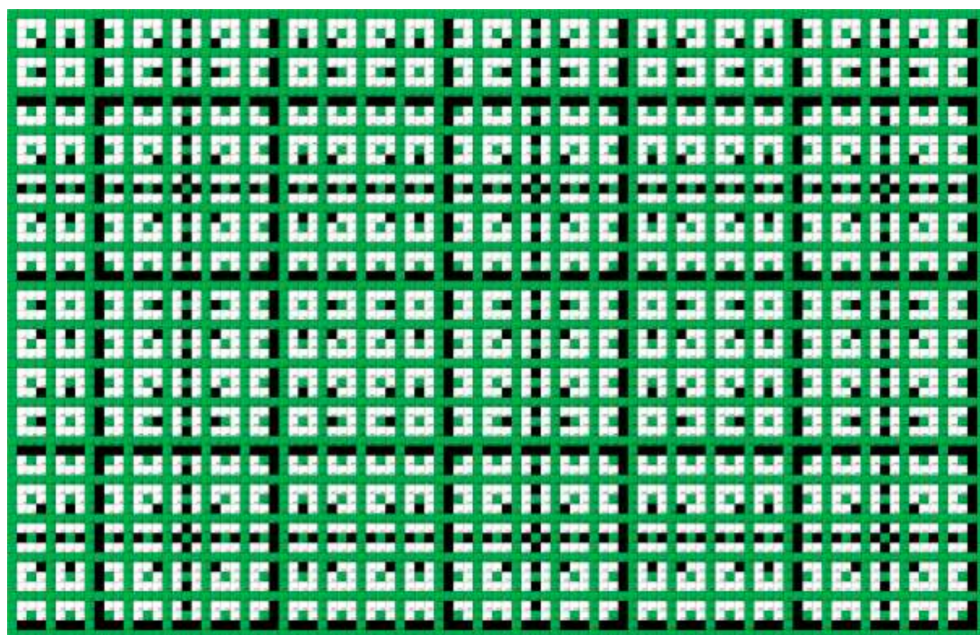
Next we use some nonconsecutive values of  $k$ . The following figure uses blue for numbers that equal to 0 modulo 6, and green for numbers that equal to 0 modulo 9.



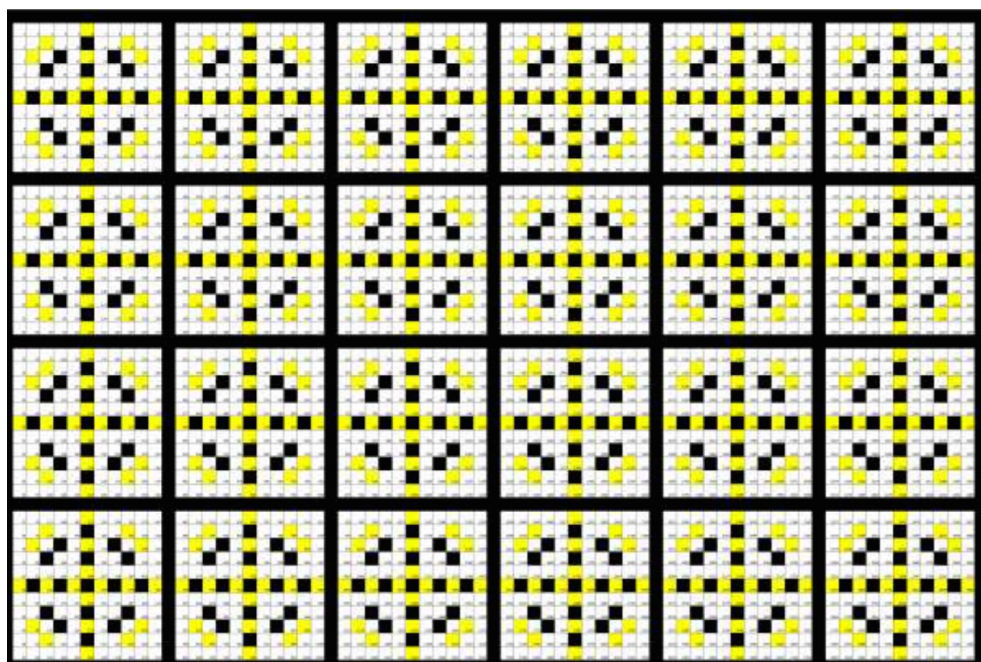
The fundamental building blocks will now be  $18 \times 18$ , as 18 is the least common multiple of the moduli used. Still the additional symmetries within the nine  $5 \times 5$  squares that make up the repeated  $17 \times 17$  squares may come as pleasant surprises. Can you find mathematical explanations for these?

In the next figure, any number that equals 0 modulo 9 is assigned to the color black, while any number that equals 0 mod 4 is assigned to the color green.

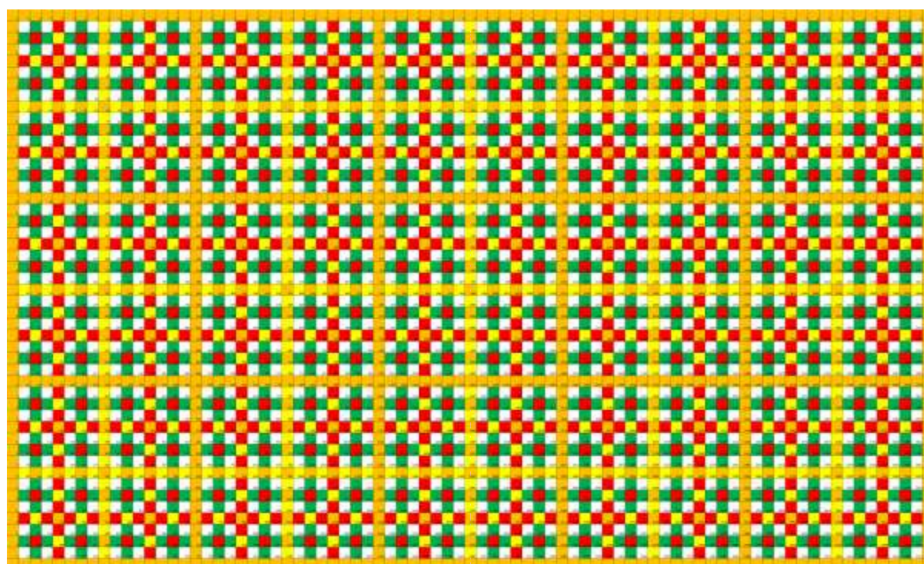




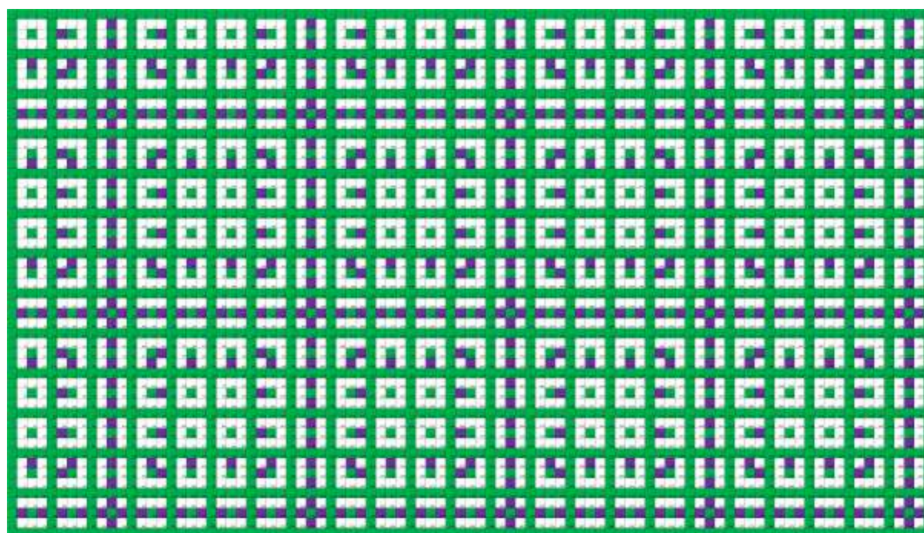
The next figure uses yellow for numbers that equal  $0 \pmod{6}$ , and black for those that equal  $0 \pmod{12}$ .



Next consider powers of 2 as moduli. Color numbers that equal 0 mod 2 green, those that equal 0 mod  $2^2$  red, those that equal 0 mod  $2^3$  yellow, and finally those that equal 0 mod  $2^4$  orange.

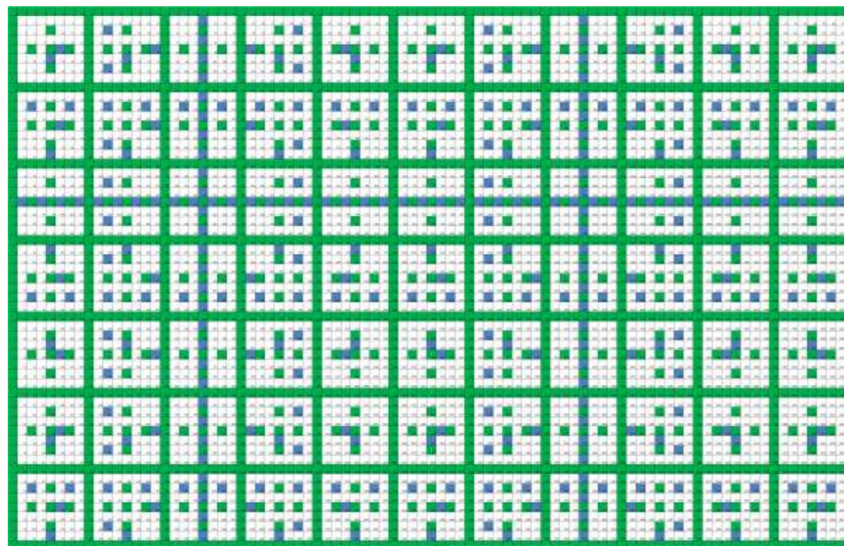


You might have noticed that non-relatively prime sets of moduli create interesting subpatterns. Consider using purple for numbers that equal 0 mod 10, and green for those that equal 0 mod 4.

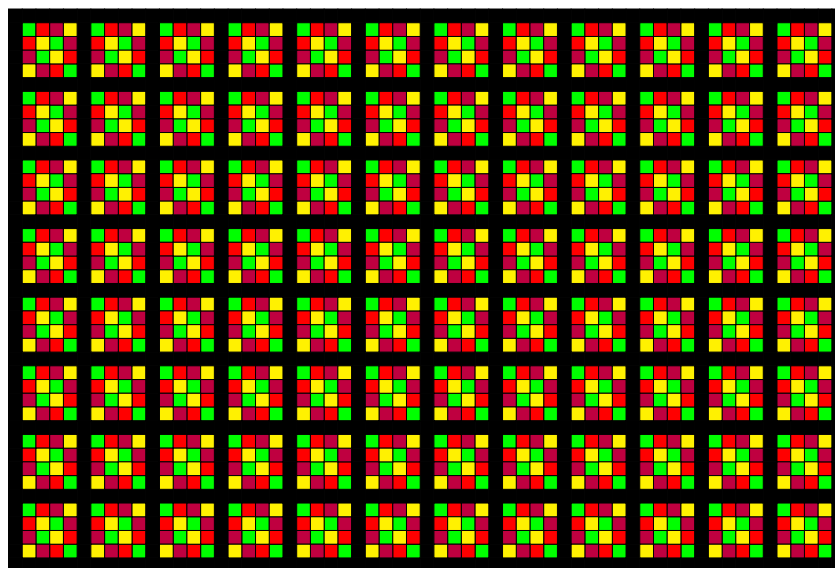




Here is another pair of non-relatively prime moduli: we use green for numbers that equal  $0 \pmod{8}$  and blue for those that equal  $0 \pmod{20}$ .



Finally, if we fix one modulus  $k$  and assign colors to cells depending on their remainder with respect to  $k$ , then all the squares can be filled in. For example, let  $0 \pmod{5}$  be black,  $1 \pmod{5}$  be green,  $2 \pmod{5}$  be red,  $3 \pmod{5}$  be purple, and  $4 \pmod{5}$  be red; the following figure is obtained.



**Conclusion**

In this note we have shown several of the symmetries that hide among the distribution of numbers in the multiplication table of positive integers. Using Excel, we created several images that display much symmetry and even fractal-like properties. All of the patterns can without much difficulty be explained by a thorough exploration of modular arithmetic and divisibility criteria. However we believe that displaying these symmetries using colors introduces a new facet of this well-known field of mathematics. These images and others created in similar ways may appeal to students of mathematics and the arts, and may lead to new collaborations. At the very least such images may, we hope, intrigue, amaze, and inspire.