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POLYHEDRAL MODELS OF THE EVERSION OF THE SPHERE; EVERSION OF THE CUB OCTAHEDRON

In some words, let us say that to return a sphere is to imagine a deformation allowing exchanging its internal face with its external face without piercing or folding its membrane.

One may press in every possible way on a balloon, the operation is impossible if one does not give him supplementary room for manoeuvre. A first condition is to authorize the surface to cross itself.

In 1957, a young researcher Stephen Smale guessed then demonstrated on one year later that it was possible to return a sphere.

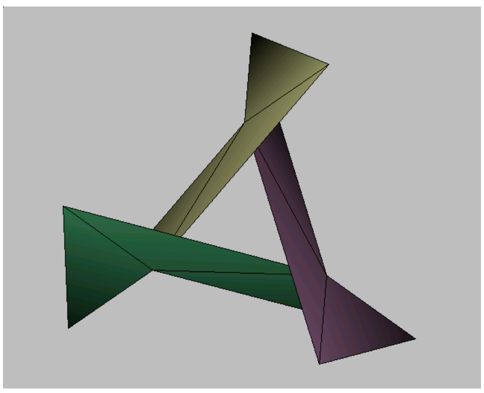
Since numerous efforts were done to understand and show this complex process. Bernard Morin, blind mathematician of the University Louis Pasteur, dried repeatedly during his career to solve this problem.

In 1989, we realized together a polyhedral version of this problem that succeeded in the eversion of the cub octahedron.

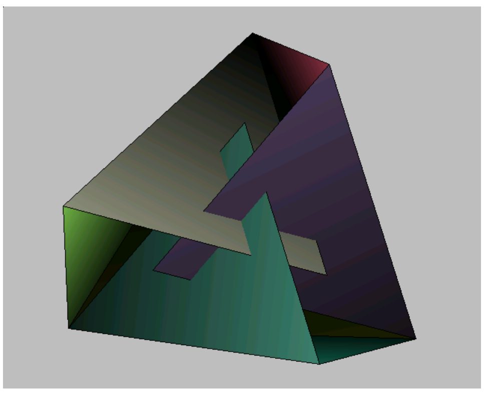
Comprehension of such a deformation requires the construction of many models; the object of this presentation is to show most significant. Thanks to Konrad Polthier's software JavaView it was possible to execute them on computer.

The capacity of this software to make cuts through the models is very useful here: it is thus possible better to understand what occurs during the deformation and to shed light on sights that would be inaccessible differently. All the models presented exist now in applets JavaView.

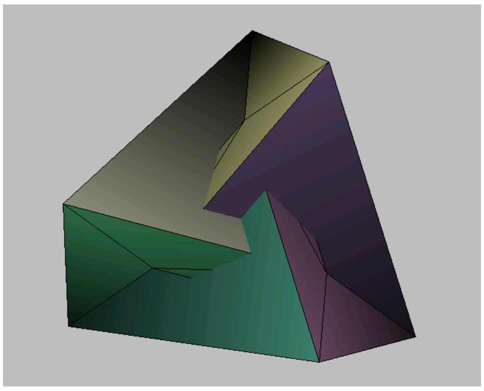
The starting point of all this work is the Boy surface imagined by Ulrich Brehm. This surface presents threefold symmetry around the axis Oz and is obtained by gluing together a Möbius strip and a disk so that the Boy surface is a representation of the real projective plane.



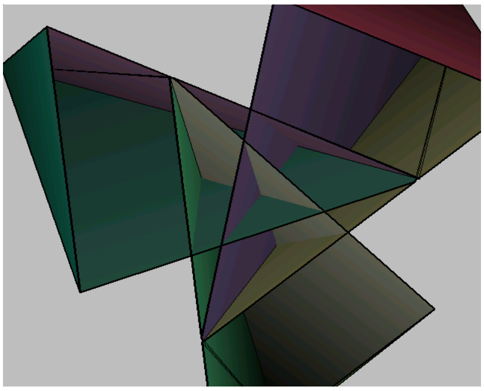
Möbius strip



Set of seven triangles homeomorphic to a disk



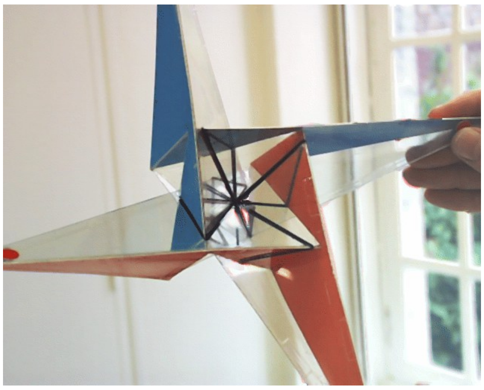
Boy surface imagined by Ulrich Brehm and Bernard Morin



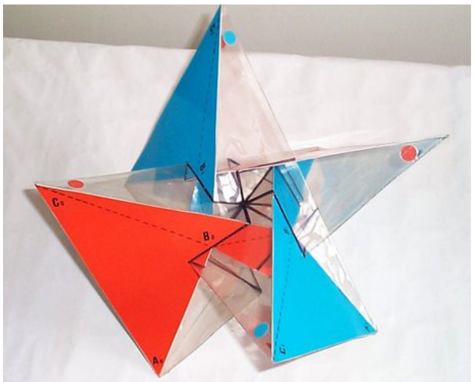
Cut under the triple point

This surface made it possible to carry out a halfway model of the eversion.

There are two types of halfway models: the opened central model and the closed central model. This last model is the most interesting to perform the eversion because it condenses in a small space the whole of the modifications (called generic modifications) that occur during the deformation.

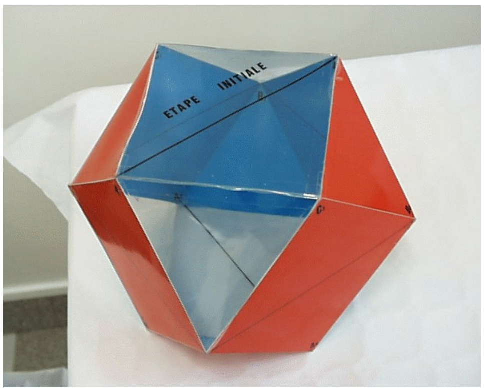


Opened central model

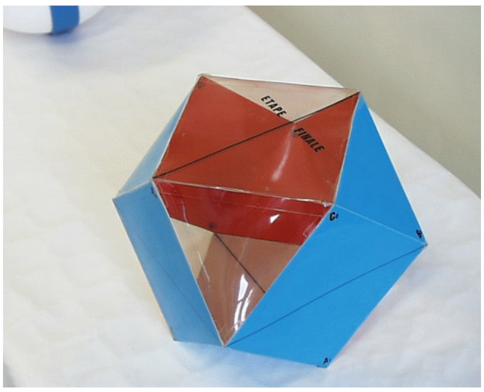


Closed central model

This complete deformation is symmetrical in time. At the beginning, at time t = - 22, one considers a cub octahedron whose outside is red and whose internal face is blue. On arrival, to time t = +22, one leads to the turned over cub octahedron whose internal face is red and the external face blue!

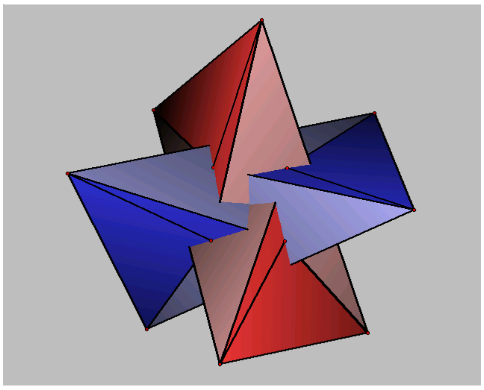


Model –22

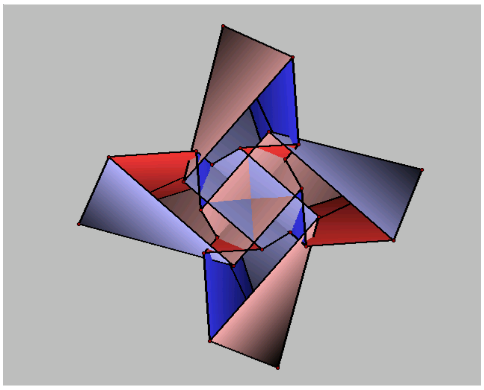


Model +22

The model obtained at time t = 0 is the closed central model. It has the characteristic to have a quadruple point, i.e. that four of its faces pass by the same point. In addition, an external observer can alternatively see one and the other of the red and blue faces of the membrane that composes it.



Model 0

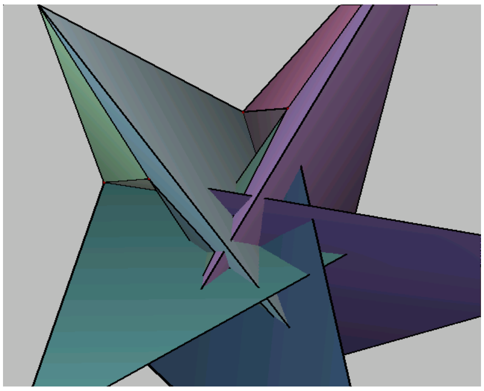


Cut under the quadruple point

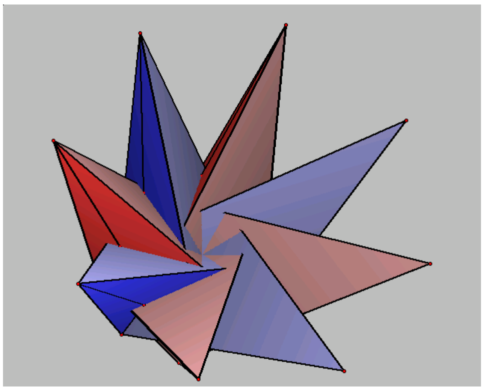
The whole deformation breaks up into a succession of elementary transformations that consist in moving a vertex along a line formed starting from edges of the polyhedron.

The Boy surface and these two central models are polyhedral examples of immersions. Several of the faces of these polyhedrons intersect themselves giving place to lines of double points, triple points as in the Boy surface and to a quadruple point as in the two central models.

The slideshow also presents an immersion of the real projective plan having fivefold symmetry as well as a generalization of the closed central model having eightfold symmetry.



Immersion of the real projective plane having a fivefold symmetry

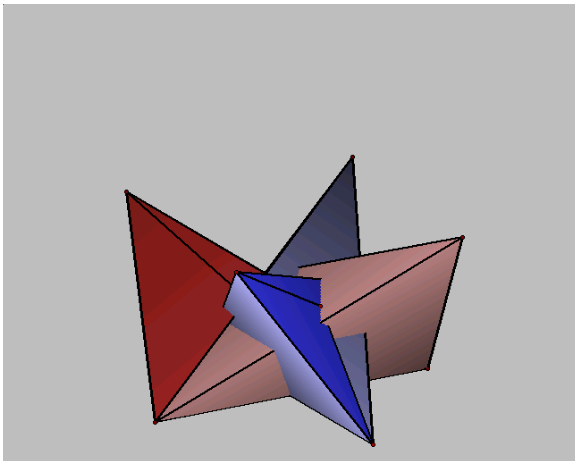


Immersion of the sphere having a eightfold symmetry

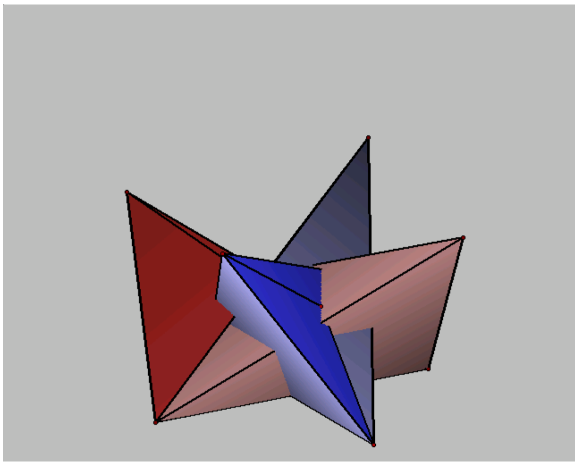
Cuts realized just under these multiple points constitute, with the pictures of the eversion of the cub octahedron, the principal parts of this slideshow.

The comprehension of the central models is essential; it contains in germ the realization of all the following models. The closed central model lies in the heart of the eversion of the cub octahedron!

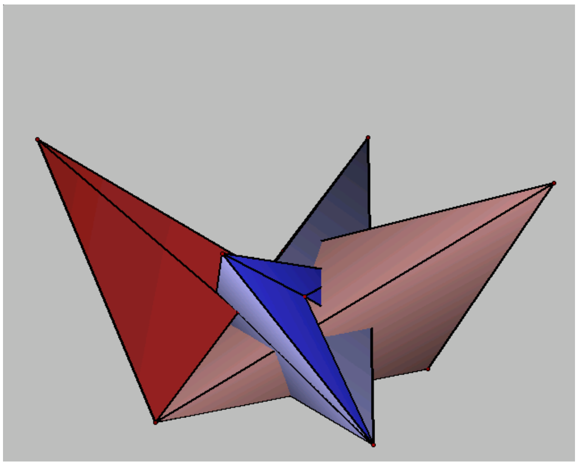
By undoing the quadruple point gradually, the triple points and the double points it is possible to leave the field of the immersions. Bernard Morin imagined how in only six steps (minimal version) it was possible to transform the closed central model into a model witch is homeomorphic to a sphere! These models correspond at stages t = 0, t = +1,..., t = + 6.



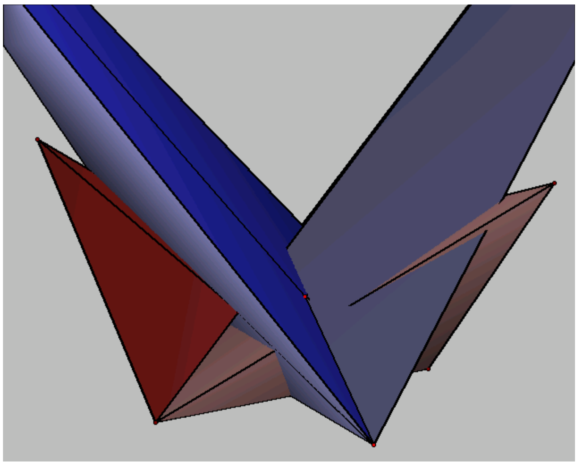
Model 0



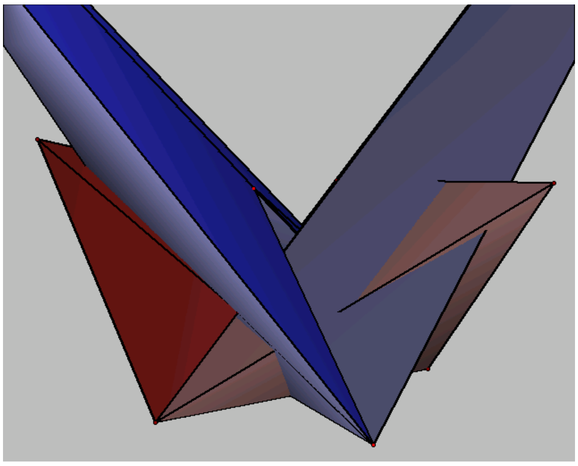
Model +1



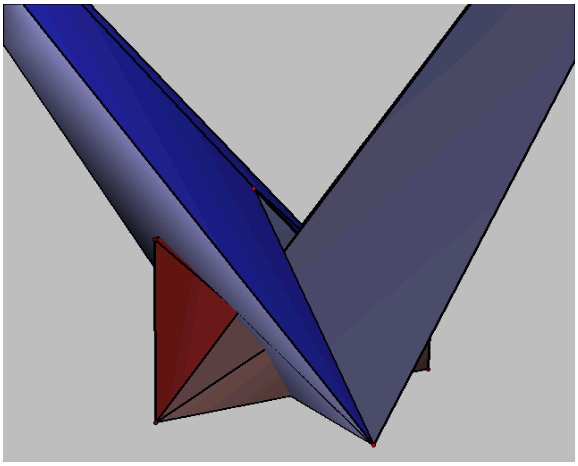
Model 2



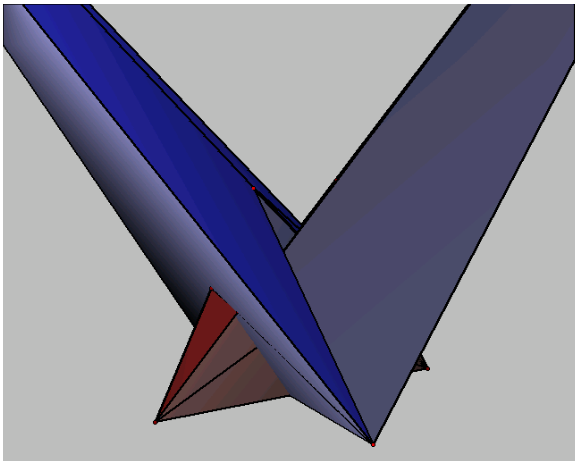
Model 3



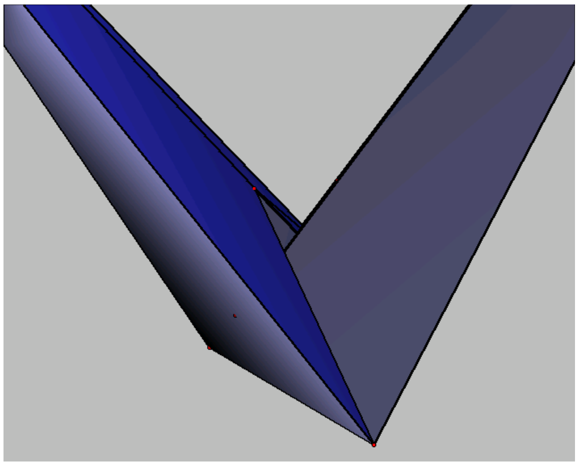
Model 4



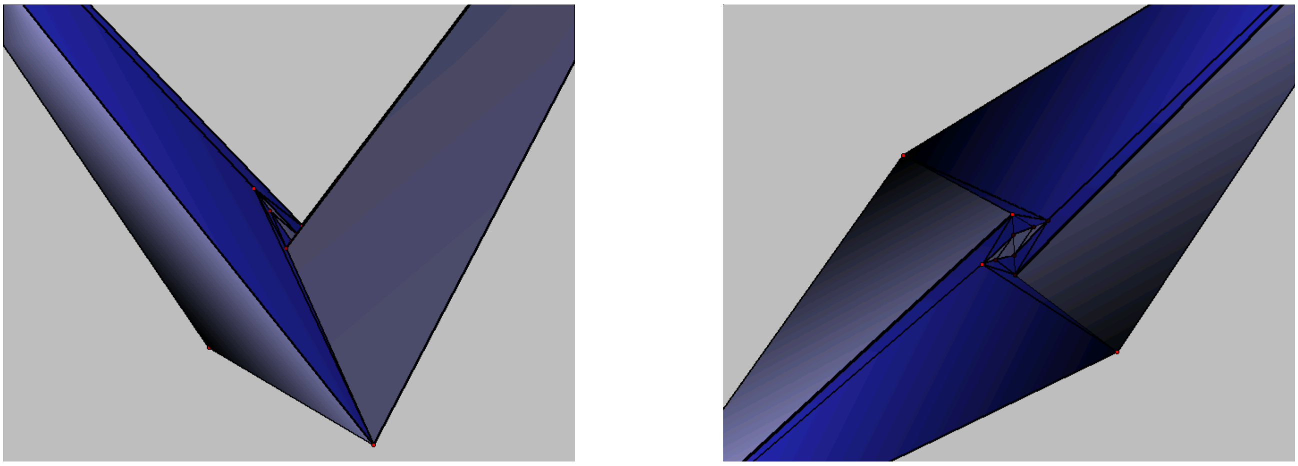
Model 4,671



Model 5

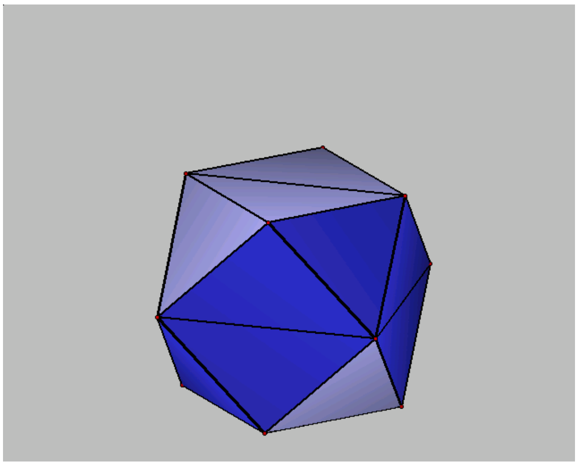


Model 5,58



Model 6

The other steps from t = +7 to t = +22 are only used to position the points to obtain the final cub octahedron.



Model +22

A similar way leads from the central model to the red octahedron (model -22).

Two models corresponding to opposite times will be practically identical except for a rotation of 90° around the axis OZ. The only difference is that what is in red on one is in blue on the other and vice versa.

When one varies continuously the time from t = -22 to t = +22, one realizes the eversion of the cub octahedron.

Eversion of the cub octahedron

Internet links:

The first description of this work was published in numbers 94 and 95 of the L'Ouvert review of the Irem of Strasbourg.
http://irem.u-strasbg.fr/irem/ouvert/som-ouv.html

The totality of the models was presented at the time of the colloquium "Arts et Mathématiques" from September 2000 at Maubeuge (France).
http://arpam.free.fr/colloque.html