

Turing Patterns, Cellular Automata, Tilings and Op-Art

Gary R. Greenfield
Department of Mathematics & Computer Science
University of Richmond
Richmond, VA 23173, USA
ggreenfi@richmond.edu

In memory of Slavik Jablan

Abstract

I provide a brief survey documenting the inclusion of cellular automata, periodic tilings and op-art in mathematical art. Then I give an overview of the history of Turing-like patterns in mathematical art. I describe a cellular automaton for producing Turing-like patterns and introduce some new variations. This leads to an open problem concerning the convergence of such patterns.

Turing and Turing-like patterns trace their origins to a proposal in 1952 by Alan Turing for modeling stable spatial patterns using systems where two or more chemicals, or morphogens, react while diffusing through a substrate at differing rates [15]. Turing hoped this would provide a biological *explanation* for vertebrate skin patterns, This did not turn out to be the case [11]. However, Turing's concept continues to influence activator-inhibitor and reaction-diffusion modeling and has been shown to occur in physical systems [2].

Young [17] proposed implementing Turing's activation-inhibition concept using a discrete cellular automaton. In 1993, in a collaboration with artist Sarah Stengle, I used Young's cellular automaton model [6] to generate source material for drawings and 3d-works (see Figure 1) that were subsequently executed by Stengle [14, pp. 34–35]. McCabe used a simpler cellular automata model based on Turing's idea wherein only one substance — representing the “pigmentation” level for each discrete cell in an array is used to produce Turing-like patterns. McCabe was thus able to generate mathematical artworks with cyclic symmetry which he called “multi-scale” Turing patterns [10]. Werth generated Turing-like patterns to use as source material for abstract acrylic paintings by using an iteration technique relying on “unsharp masking” to solve a system of reaction-diffusion partial differential equations [16]. More recently, I have used Young's cellular automaton formulation to explore Turing-like patterns created by letting parameters that were previously fixed vary from cell to cell; to investigate Turing-like patterns with transparency effects; and to incorporate alternative distance metrics [7]. Simultaneously, Schwehm reformulated McCabe's algorithm so that it could be used to generate what he calls Turing-McCabe patterns at rates up to twenty frames per second so that they could be incorporated into interactive performance art [12].

It has been understood since antiquity that when periodic boundary conditions are imposed on patterns, they can be used for tiling the plane as well as creating op-art effects [9]. More to the point, Turing patterns with periodic boundary conditions [4, 10], patterns that aspire to op-art [5, 8, 13] and patterns involving cellular automata [1] [3, pp. 38–39] are prominent in mathematical art.

Figure 2 shows a periodic tiling I made using a tile generated by compositing images obtained from a reaction-diffusion simulation that was based on remote-sensing virtual agents [4]. Figure 3 shows an op-art piece I made from a cellular morphogenesis simulation [4]. With reference to my recent artwork generated using variations of Young's cellular automaton for Turing-like patterns [7], Figure 4 shows a periodic op-art pattern I created. Figure 5 shows three periodic tiles I created by using different distance metrics. Here, I continue to explore variations of Young's cellular automata for art making purposes.



Figure 1: *Left: “Greenfield Drawing”, Sarah Stengle, 1994. Ink on transparent vellum, sheet size 11" × 14", image size 7" × 10". Right: “Reaction Diffusion Glass Book”, 1994. Gold leaf on glass in a painted wooden frame with brass hinges, 10" × 10" × 8". Reaction diffusion modeled by Gary Greenfield, book by Sarah Stengle.*

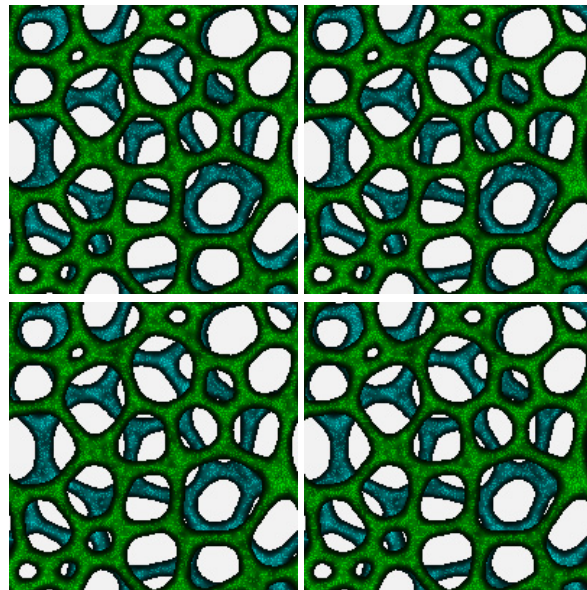


Figure 2: *Tiling produced from a periodic reaction-diffusion simulation.*

Young’s starting point is an activator-inhibitor theory due to Swindale that provides the generalized diffusion equation,

$$\frac{\partial M}{\partial t} = \nabla \cdot D \cdot \nabla M - KM + Q$$

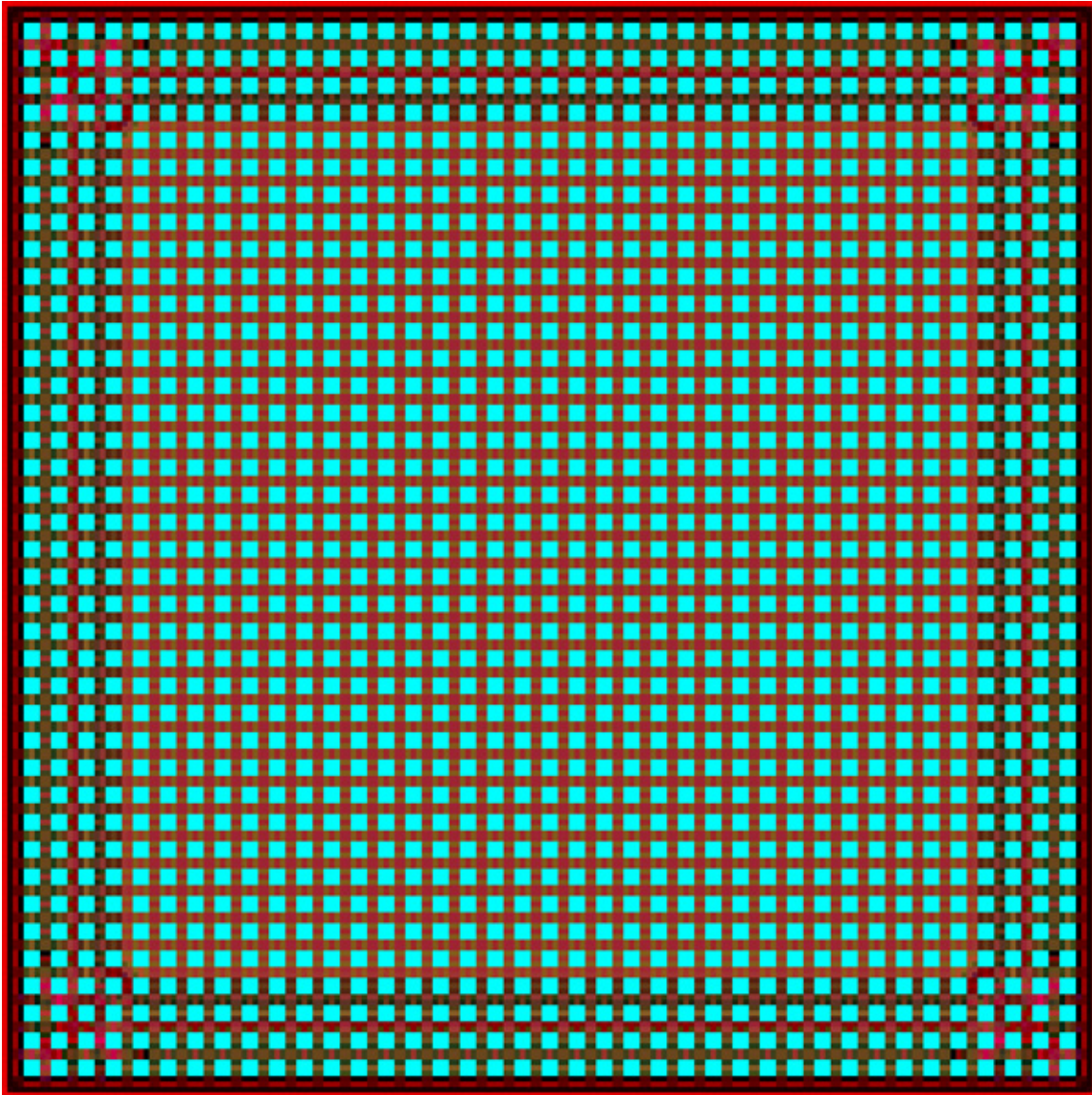


Figure 3: *“It’s Not That Simple, II”*, Gary Greenfield, 2009.

where $M = M(\vec{r}, t)$ is the morphogen (either activator or inhibitor) concentration, and the terms on the right are diffusion, first-order chemical transformation, and production, respectively. A cell producing both an activator and inhibitor morphogen at a constant rate, each diffusing according to this equation, induces a “morphogenic field” $w(R)$ a distance R from the cell.

Using this model, the cellular automaton Young proposes consists of a two-dimensional array of cells. Cells are designated to be either differentiated cells (DCs) or undifferentiated cells (UCs). DCs produce both an activator and an inhibitor which are modeled so as to be updated in accordance with the Swindale equation. The activator stimulates nearby UCs to become DCs and the inhibitor discourages faraway DCs from remaining so. More precisely, consider the annulus of cells surrounding a DC with inner radius r_1 and outer radius r_2 , where $0 < r_1 < r_2$. All cells within distance r_1 of the DC receive w_1 units of activator, where $w_1 > 0$, and all cells whose distance is between r_1 and r_2 of the DC receive w_2 units of inhibitor, where $w_2 < 0$. A cell’s status as a DC or UC is determined by the sum of all the morphogens it receives

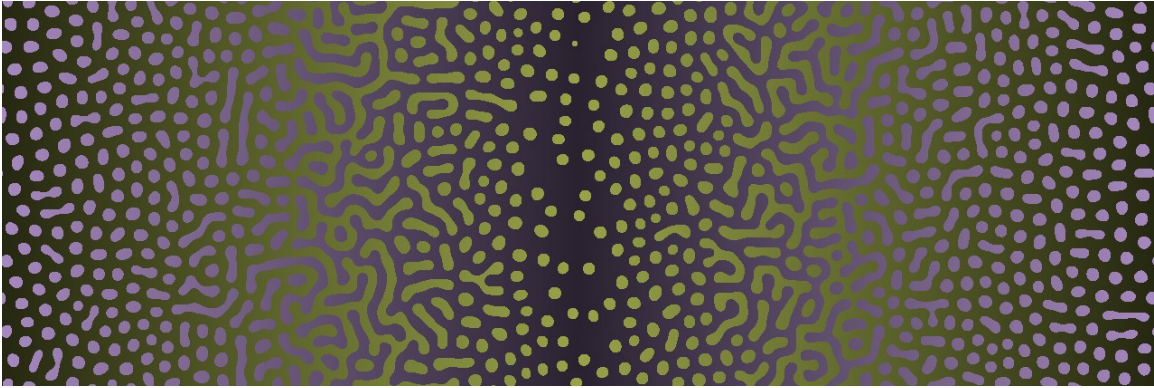


Figure 4: “Turing Gradient Pattern #30423”, Gary Greenfield, 2015.

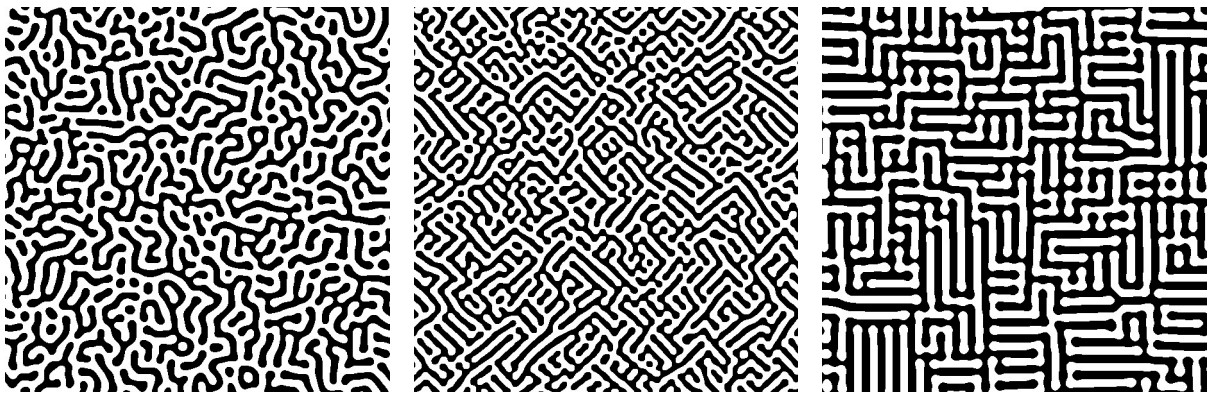


Figure 5: “Turing Metric Trio”, Gary Greenfield, 2016.

from all the nearby DCs.

For convenience, normalize w_1 to +1.0. Using periodic boundary conditions for cell neighborhoods so that after every iteration, or update, the pattern will seamlessly tile the plane, each update requires execution of the following three-pass procedure.

Pass One. Zero out all cells’ w values.

Pass Two. For each DC encountered add the appropriate w_1 or w_2 value to the w value of each cell in the annulus defined above.

Pass Three. Assign cells to be DCs or UCs according as their accumulated w value is positive or negative.

Note that if no cells change state during the final pass an *equilibrium* state for the pattern has been reached and no further updates are warranted. For initialization purposes, a randomly chosen set of cells (typically consisting of around 15% of the total) are designated to be DCs.

Figure 6 shows an example of the kinds of effects one can obtain by adding “directionality” to the basic model by treating the cells that spread morphogens as if they are at the edge of the activation-inhibition neighborhood rather than at the center. That is, by defining the neighborhood of a DC whose coordinates are (x_0, y_0) to be the set of cells whose coordinates (x, y) satisfy $x_0 \leq x \leq x_0 + L$ and $y_0 \leq y \leq y_0 + L$, where $L > r_2$ is constant, during the second pass we only increment the w values of cells that lie in the intersection of this neighborhood and the annulus surrounding the DC.

Figure 7 gives examples where there are one or more regions with differing activation-inhibition characteristics. That is w_2 takes on different values depending on where in the array the DC is located. Figure 8

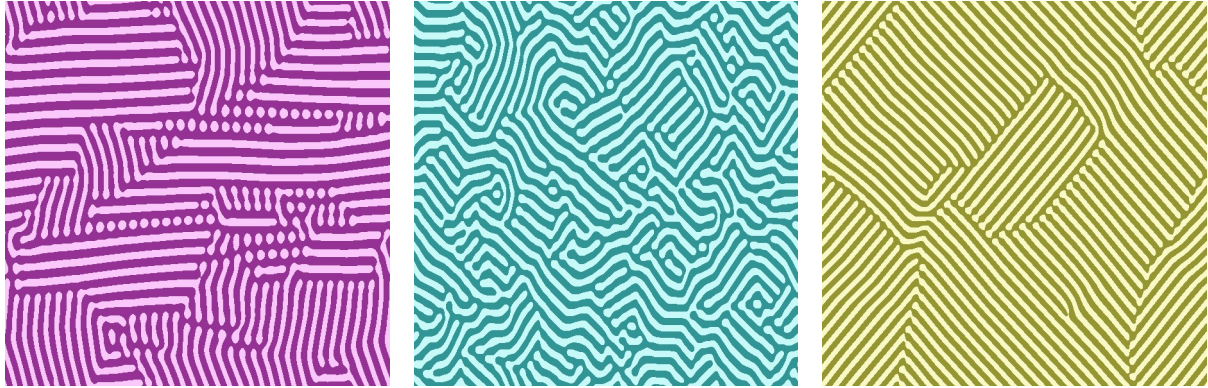


Figure 6: Turing patterns under different metrics with directionality added.

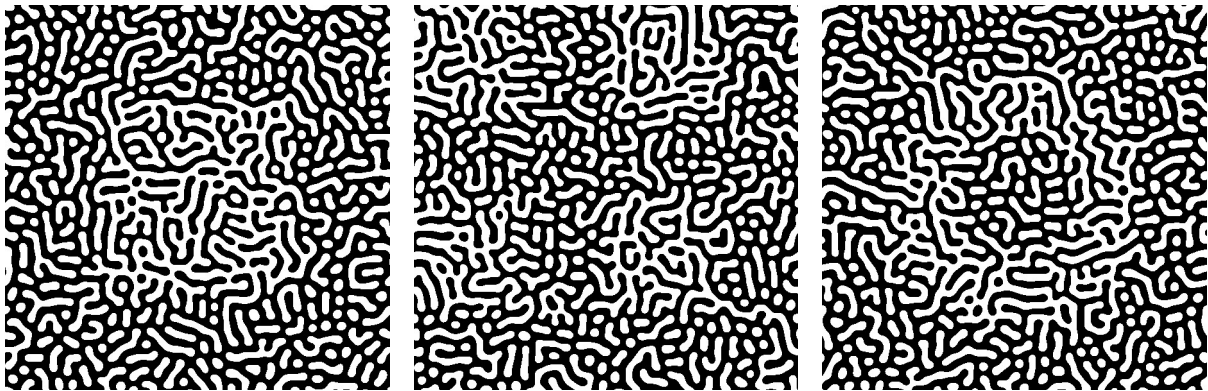


Figure 7: Examples of Turing patterns where there are regions with different activation-inhibition characteristics.

shows a tiling induced using the middle pattern of Figure 7. These patterns need to be viewed at a distance.

Finally, I consider the “equilibrium problem” for some of the variations of Young’s automata that I have studied. To the naked eye, a stable pattern always emerges from Young’s model after just a few iterations of the update algorithm. In most situations, after a few hundred iterations, the pattern reaches a true equilibrium meaning no cell changes state if additional iterations of the update algorithm are performed. Exactly why this seems to occur is not clear. However, when I was experimenting with activation-inhibition “half-neighborhoods” of a differentiated cell at location (x_0, y_0) defined by

$$\{(x, y) : x > x_0 \text{ and } d_m((x, y), (x_0, y_0)) < L\}$$

where d_m is the metric and L is the radius of the ball centered at (x_0, y_0) under d_m the situation became more interesting. Figure 9 shows a pattern under the Euclidean metric that after approximately 500 iterations becomes stable to the naked eye but then upon further iterations of the update algorithm “slide” across the underlying toroidal grid in such a way that exactly 50990 cells change state at each iteration. Moreover, using the same initial seeding pattern but using either the taxicab or supremum metrics it does not appear that the pattern ever even reaches a stable configuration! Figure 10 shows the supremum pattern starting from the same initial configuration used for Figure 9 after 500, 1000 and 1500 iterations while Figure 11 shows the taxicab pattern starting from the same initial configuration after 1000, 2000, and 3000 iterations.

A simple combinatorics argument shows that a Turing-like pattern that does not reach equilibrium must eventually oscillate: If there are N cells in the array, then there are 2^N possible DC and UC state configurations, and since the w values for cells are zeroed during Pass One, by the Pigeon Hole Principle

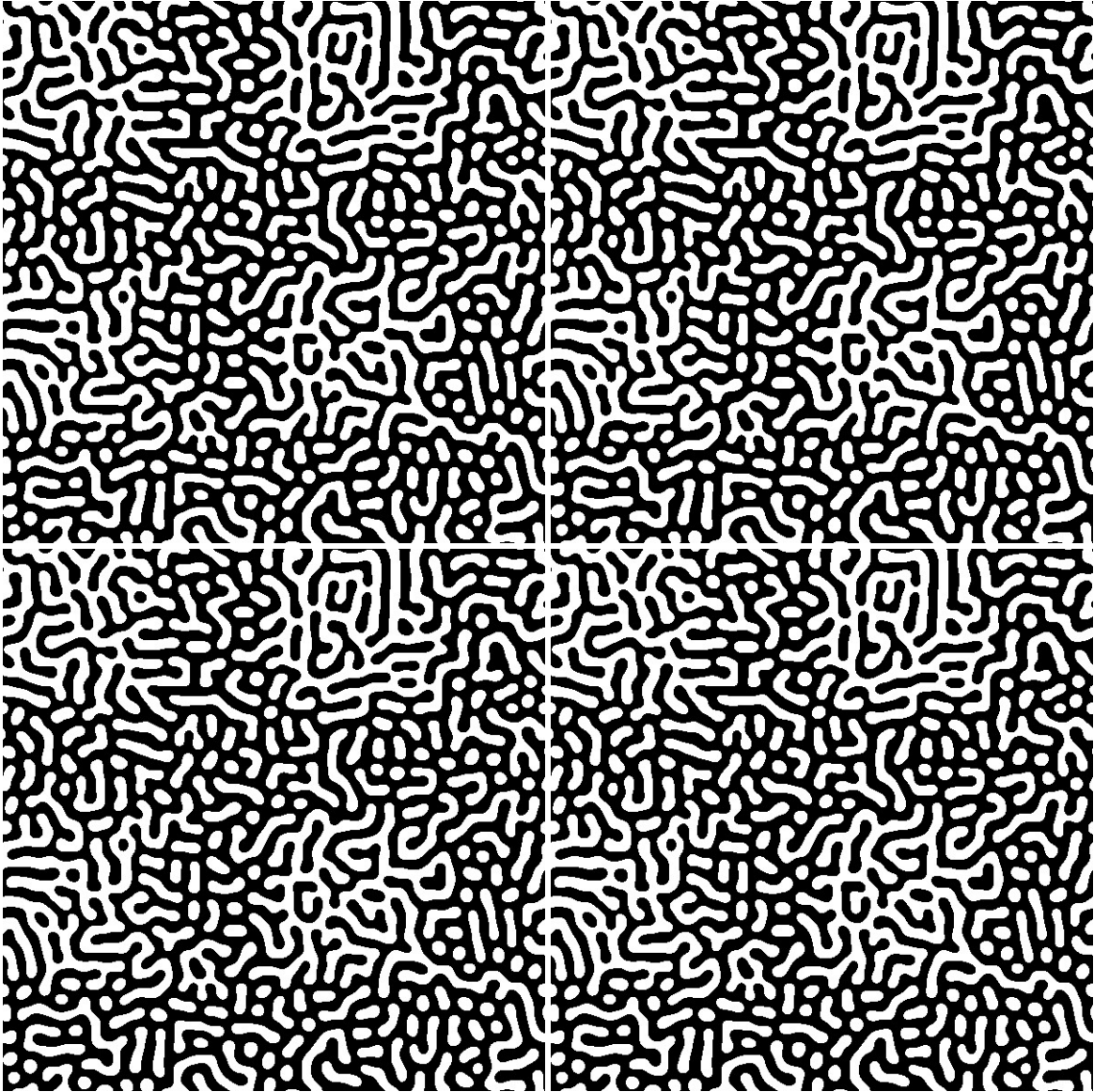


Figure 8: *Tiling produced from one of our multiple-region patterns.*

after $1 + 2^N$ iterations a configuration must have appeared twice signaling that oscillation has begun. It is an open, and we presume hard if not intractable, problem to determine whether an initial configuration will reach equilibrium or will oscillate.

To summarize, and in conclusion, I have given a brief, non-exhaustive survey of the presence of cellular automata, periodic tilings and op-art in mathematical art and I have given an overview of the history of what are variously called Turing patterns, Turing-like patterns and Turing-McCabe patterns in mathematical art. Further, I have introduced some new variations of a Turing-like pattern generating cellular automaton due to Young and identified an open problem concerning the convergence of the patterns for these variations.

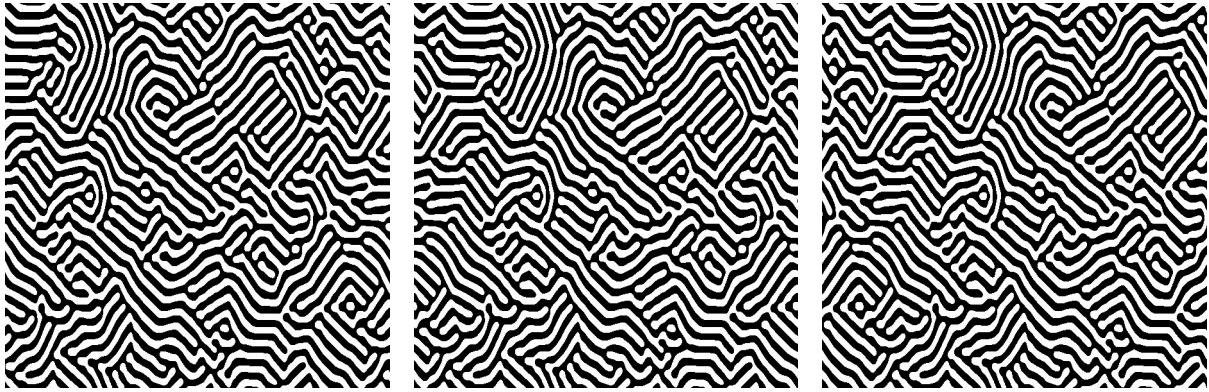


Figure 9 : A sliding pattern found using Euclidean “half-neighborhoods” such that exactly 50990 cells change state at each iterations. The pattern is shown after 500, 530 and 560 iterations.

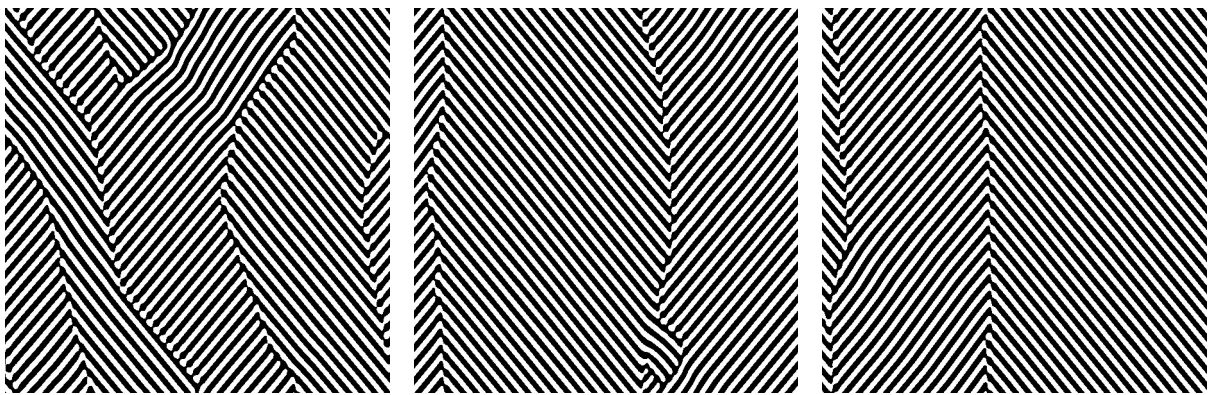


Figure 10 : The slow convergence of the same starting configuration as the previous figure under the supremum metric after 500, 1000 and 1500 iterations.

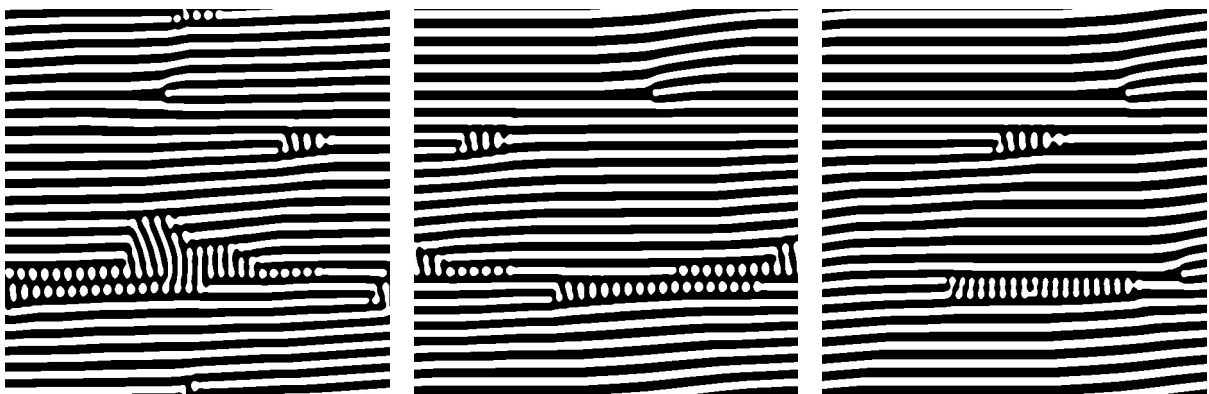


Figure 11 : The even slower convergence of the same starting configuration as the previous figure under the taxicab metric after 1000, 2000 and 3000 iterations.

References

- [1] Robert Bosch and Julia Olivieri, Designing Game of Life mosaics with integer programming, *J. of Mathematics and the Arts*, vol. 8, nos. 3-4, 2014, pp. 120–132.

- [2] V. Castets, E. Dulos, J. Boissonade and P. De Kepper, Experimental evidence of a sustained standing Turing-type nonequilibrium chemical pattern, *Physical Review Letters*, vol. 64, 1990, pp. 2953–2956.
- [3] Robert Fathauer and Nathan Selikoff, eds., *2016 Joint Mathematics Meetins Exhibition of Mathematical Art*, Tessellations Publishing, Phoenix, AZ, 2016.
- [4] Gary Greenfield, Abstract overlays using a transport network model, *Bridges 2011 Conference Proceedings*, C. Sequin and R. Sarhangi, eds., Tessellations Publishing, Phoenix, AZ, 2011, pp. 45–50.
- [5] Gary Greenfield, Bio-op art, *Proceedings of ISAMA 2010, Ninth Interdisciplinary Conference of the International Society of the Arts, Mathematics, and Architecture*, E. Akleman and N. Friedman, eds., Hyperseeing, Summer 2010, pp. 31–38.
- [6] Gary Greenfield, Case Study: A Sculptor - Programmer Collaboration, Technical Report TR-93-03, Mathematics and Computer Science, University of Richmond, 1993.
- [7] Turing-like patterns from cellular automata, *Bridges 2016 Conference Proceedings*, E. Torrence, B. Torrence, C. Séquin, D. McKenna, K. Fenyvesi and R. Sarhangi (Eds.), Tessellations Publishing, Phoenix, AZ, 2016, pp. 151–158.
- [8] Tiffany Inglis and Craig Kaplan, Animating line-based op art, *Bridges 2013 Conference Proceedings*, George W. Hart and Reza Sarhangi, eds., Tessellations Publishing, Phoenix, AZ, 2013, pp. 159–166.
- [9] Slavik Jablan and Ljiljana Radovic, Do you like paleolithic op-art?, *Kybernetes*, vol. 40, no. 9, 2011, pp. 1045–1054.
- [10] Jonathan McCabe, Cyclic symmetric multi-scale Turing patterns, in *Bridges 2010 Conference Proceedings*, George W. Hart and Reza Sarhangi, eds., Tessellations Publishing, Phoenix, AZ, 2015, pp. 387–390.
- [11] J. D. Murray, Why are there no three-headed monsters? Mathematical modeling in biology, *Notices of the AMS*, vol. 59, no. 6, June/July 2012, pp. 785–795.
- [12] Markus Schwehm, A fast algorithm for creating Turing-McCabe patterns, *Bridges 2016 Conference Proceedings*, E. Torrence, B. Torrence, C. Séquin, D. McKenna, K. Fenyvesi and R. Sarhangi (Eds.), Tessellations Publishing, Phoenix, AZ, 2016, pp. 431–434.
- [13] John Sharp, Linkages to op-art, *Bridges 2010 Conference Proceedings*, Reza Sarhangi and John Sharp, eds., Tarquin Publications, London, 2010, pp. 497–502.
- [14] Sarah Stengle, *Without Me: Diverse Themes*, Exhibition catalog from Zoellner Art Center, Lehigh University, Bethlehem, PA, 2001.
- [15] A. Turing, The chemical basis of morphogenesis, *Philosophical Transactions of the Royal Society of London*, vol. B, no. 237, 1952, pp. 37–72.
- [16] Andrew Werth, Turing patterns in Photoshop, in *Bridges 2015 Conference Proceedings*. Kelly Delp, Craig S. Kaplan, Douglas McKenna and Reza Sarhangi, eds., Tessellations Publishing, Phoenix, AZ, 2015, pp. 459–462.
- [17] D. A. Young, A local activator-inhibitor model of vertebrate skin patterns, in *Theory & Application of Cellular Automata*, S. Wolfram (ed.), World Scientific, 1986, pp. 320–327.